refract ray

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here is comment in `Triangulate.refractRay()`
of project `underwater-camera-calibration`

```
# - 'rayDir' is the vector of the incoming ray
# - 'planeNormal' is the plane normal of the refracting interface
# - 'n1' is the refraction index of the medium the ray travels >FROM<
# - 'n2' is the refractio index of the medium the ray travels >TO<
```

table of symbol

SMYBOL	MEANING
v_1	unit direction vector (FROM)
v_2	unit direction vector (TO)
n	unit normal vector of interface plane (FROM / TO)
n_1	refraction index (FROM)
n_2	refraction index (TO)
θ_1	angle (FROM)
θ_2	angle (TO)

analysis

We should notice that n is opposed to the incoming ray direction v_1:

$$v_1^T n < 0, v_1^T v_1 = v_2^T v_2 = n^T n = 1$$

Now that we know all the parameters except v_2,
we need derive the expression of v_2 with other variables v_1, n, n_1, n_2

reconstruct vector

and notice cross product matrix $n \times n \times = nn^T - I$ and $n^T n = 1$:

$$
\begin{aligned}
|v_1| \cos\theta_1 &\equiv \frac{-v_1^T n}{|n|} = -v_1^T n \\
|v_1|^2 \sin^2\theta_1 &\equiv \left| v_1 - (-n)\frac{(-n^T v_1)}{n^T n} \right|^2 = \left| v_1 - \frac{nn^T}{n^T n} v_1 \right|^2 = |(I - nn^T)v_1|^2 \\
&= |-n \times (n \times v_1)|^2 = v_1^T (I - nn^T)^T (I - nn^T) v_1 \\
&= v_1^T (I - nn^T)(I - nn^T) v_1 \\
&= v_1^T (I - nn^T) v_1 = v_1^T [-n \times n \times v_1] v_1 \\
&= v_1^T v_1 - [v_1^T n]^2 \\
v_1 &\equiv nn^T v_1 + (I - nn^T) v_1 = n(n^T v_1) - n \times (n \times v_1) \\
&= (-n)|v_1| \cos\theta_1 + \frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|} |v_1| \sin\theta_1
\end{aligned}
$$

Use $v_1^T v_1 = 1$:

$$
\begin{aligned}
\sin\theta_1 &= \sqrt{\frac{v_1^T v_1 - [v_1^T n]^2}{v_1^T v_1}} = \sqrt{1 - [v_1^T n]^2} \\
v_1 &= (-n)\cos\theta_1 + \frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|} \sin\theta_1
\end{aligned}
$$

For the same reason, $v_2^T v_2 = 1$,
and v_1, v_2, n in the same plane: $n \times v_1, n \times v_2$ are linear related,
and $v_1^T n < 0, v_2^T n < 0$:

$$
\frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|} = \frac{[-n \times (n \times v_2)]}{|-n \times (n \times v_2)|}
$$

We want to reconstruct v_2 with n, v_1, similarly:
here is decomposition of orthogonal basis,
because $n^T [n \times n \times v_1] = n^T [nn^T - I] v_1 = 0^T v_1 = 0$

$$
\begin{aligned}
v_2 &= (-n)\cos\theta_2 + \frac{[-n \times (n \times v_2)]}{|-n \times (n \times v_2)|} \sin\theta_2 \\
&= (-n)\cos\theta_2 + \frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|} \sin\theta_2
\end{aligned}
$$

Snell's Law

From previous formula, only things are missing to reconstruct v_2 is $\sin\theta_2, \cos\theta_2$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

We want to express θ_1, θ_2 with parameters that we know v_1, n, n_1, n_2,

$$\begin{aligned}
\cos\theta_1 &= -v_1^T n \\
\sin\theta_1 &= \sqrt{\frac{v_1^T v_1 - [v_1^T n]^2}{v_1^T v_1}} = \sqrt{1 - [v_1^T n]^2} \\
\sin\theta_2 &= \frac{n_1}{n_2} \sin\theta_1 = \frac{n_1}{n_2} \sqrt{1 - [v_1^T n]^2} \\
\cos\theta_2 &= \sqrt{1 - (\frac{n_1}{n_2})^2 (1 - [v_1^T n]^2)}
\end{aligned}$$

Represent $\frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|}$ with v_1, n:

$$\begin{aligned}
\frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|} &= \frac{v_1 + n\cos\theta_1}{\sin\theta_1} \\
\frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|} \sin\theta_2 &= v_1 \frac{\sin\theta_2}{\sin\theta_1} + n \frac{\sin\theta_2}{\sin\theta_1} \cos\theta_1 \\
&= v_1 (\frac{n_1}{n_2}) - n n^T v_1 (\frac{n_1}{n_2}) \\
&= (I - n n^T) v_1 (\frac{n_1}{n_2}) \\
&= -n \times (n \times v_1)(\frac{n_1}{n_2})
\end{aligned}$$

Eventually, represent v_2 with v_1, n, n_1, n_2

$$\begin{aligned}
v_2 &= (-n)\cos\theta_2 + \frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|} \sin\theta_2 \\
&= n \left[-\sqrt{1 - (\frac{n_1}{n_2})^2 (1 - [v_1^T n]^2)} \right] + v_1 (\frac{n_1}{n_2}) - n \left[n^T v_1 (\frac{n_1}{n_2}) \right] \\
&= n \left[(\frac{n_1}{n_2}) [-n^T v_1] - \sqrt{1 - (\frac{n_1}{n_2})^2 (1 - [v_1^T n]^2)} \right] + v_1 (\frac{n_1}{n_2})
\end{aligned}$$

Consider all the possible values of v1

If the family of vector v_1 are always on the same plane,
the unit normal vector of this plane is π, always holds:

$$\pi^T v_1 = 0$$

We want to prove always exist $A, B \neq 0$ for any v_1 that holds $\pi^T v_1 = 0$, make sure

$$
\begin{aligned}
(A\pi + Bn)^T v_2 &= 0 \\
&= A[\pi^T n]\left[(\frac{n_1}{n_2})[-n^T v_1] - \sqrt{1 - (\frac{n_1}{n_2})^2 (1 - [v_1^T n]^2)} \right] \\
&- B\sqrt{1 - (\frac{n_1}{n_2})^2 (1 - [v_1^T n]^2)} \\
&= A[\pi^T n](\frac{n_1}{n_2})[-n^T v_1] \\
&- \left[A[\pi^T n] + B \right] \sqrt{1 - (\frac{n_1}{n_2})^2 (1 - [v_1^T n]^2)} \\
\frac{A[\pi^T n] + B}{A[\pi^T n]} &= \frac{(\frac{n_1}{n_2})[-n^T v_1]}{\sqrt{1 - (\frac{n_1}{n_2})^2 (1 - [v_1^T n]^2)}}
\end{aligned}
$$

So, constant A, B don't exist when v_1 keeps changing

triangulate

Consider replace symbol of refraction:

- replace v_1 with r to represent unit direction vector (FROM)
- replace v_2 with r' to represent unit direction vector (TO)

Then the formula of r' becomes:

$$r' = n\left[(\frac{n_1}{n_2})[-n^T r] - \sqrt{1-(\frac{n_1}{n_2})^2(1-[r^T n]^2)}\right] + r(\frac{n_1}{n_2})$$

symbol table

SMYBOL	MEANING
r'_1	unit direction vector of ray 1 in water
r'_2	unit direction vector of ray 2 in water
I_1	intersection point of interface plane and ray 1
I_2	intersection point of interface plane and ray 2
M_1	closest point of ray 1 to ray 2
M_2	closest point of ray 2 to ray 1
M	mid point of M_1 and M_2
k_1	scalar factor from I_1 to M_1
k_2	scalar factor from I_2 to M_2

Our goal is to express M with what we know r'_1, r'_2, I_1, I_2

analysis

Defination of M_1, M_2,
the closest point pair $M_1 - M_2$ must be perpendicular to r'_1, r'_2:

$$
\begin{aligned}
{r'}_1^T(M_1 - M_2) = 0 \\
{r'}_2^T(M_1 - M_2) = 0
\end{aligned}
$$

Definition of k_1, k_2:

$$
\begin{aligned}
k_1 r'_1 \equiv M_1 - I_1 \\
k_2 r'_2 \equiv M_2 - I_2
\end{aligned}
$$

Replace M_1, M_2 with unknow k_1, k_2
and what we know r'_1, r'_2, I_1, I_2,
To solve k_1, k_2 firstly

$$
\begin{aligned}
{r'}_1^T\left([I_1 - I_2] + k_1 r'_1 - k_2 r'_2\right) = 0 \\
{r'}_2^T\left([I_1 - I_2] + k_1 r'_1 - k_2 r'_2\right) = 0
\end{aligned}
$$

It is equivalent to

$$
\begin{aligned}
[{r'}_1^T r'_1]k_1 - [{r'}_1^T r'_2]k_2 = -{r'}_1^T[I_1 - I_2] \\
-[{r'}_2^T r'_1]k_1 + [{r'}_2^T r'_2]k_2 = {r'}_2^T[I_1 - I_2]
\end{aligned}
$$

In matrix form

$$
\begin{pmatrix} [{r'}_1^T r'_1] & -[{r'}_1^T r'_2] \\ -[{r'}_1^T r'_2] & [{r'}_2^T r'_2] \end{pmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{pmatrix} -{r'}_1^T[I_1 - I_2] \\ {r'}_2^T[I_1 - I_2] \end{pmatrix}
$$

With Cramer's rule

$$k_1 = \frac{\begin{vmatrix} -{r'}_1^T[I_1 - I_2] & -[{r'}_1^T r'_2] \\ {r'}_2^T[I_1 - I_2] & [{r'}_2^T r'_2] \end{vmatrix}}{\begin{vmatrix} [{r'}_1^T r'_1] & -[{r'}_1^T r'_2] \\ -[{r'}_1^T r'_2] & [{r'}_2^T r'_2] \end{vmatrix}} = \frac{[-{r'}_2^T r'_2 {r'}_1^T + {r'}_2^T r'_1 {r'}_2^T][I_1 - I_2]}{1 - [{r'}_1^T r'_2]^2}$$

$$= {r'}_2^T \left[\frac{1}{1 - [{r'}_1^T r'_2]^2} \left(r'_1 {r'}_2^T - r'_2 {r'}_1^T \right) [I_1 - I_2] \right]$$

$$k_2 = \frac{\begin{vmatrix} [{r'}_1^T r'_1] & -{r'}_1^T[I_1 - I_2] \\ -[{r'}_1^T r'_2] & {r'}_2^T[I_1 - I_2] \end{vmatrix}}{\begin{vmatrix} [{r'}_1^T r'_1] & -[{r'}_1^T r'_2] \\ -[{r'}_1^T r'_2] & [{r'}_2^T r'_2] \end{vmatrix}} = \frac{[{r'}_1^T r'_1 {r'}_2^T - {r'}_1^T r'_2 {r'}_1^T][I_1 - I_2]}{1 - [{r'}_1^T r'_2]^2}$$

$$= {r'}_1^T \left[\frac{1}{1 - [{r'}_1^T r'_2]^2} \left(r'_1 {r'}_2^T - r'_2 {r'}_1^T \right) [I_1 - I_2] \right]$$

expression of M

Definition of M is

$$M \equiv \frac{M_1 + M_2}{2} = \frac{I_1 + I_2}{2} + \frac{k_1 r'_1 + k_2 r'_2}{2}$$

$$= \frac{I_1 + I_2}{2} + \frac{1}{2} \left(r'_1 {r'}_2^T + r'_2 {r'}_1^T \right) \left[\frac{1}{1 - [{r'}_1^T r'_2]^2} \left(r'_1 {r'}_2^T - r'_2 {r'}_1^T \right) [I_1 - I_2] \right]$$

$$= \frac{I_1 + I_2}{2} + \frac{1}{2} \frac{1}{1 - [{r'}_1^T r'_2]^2} \left[\left(r'_1 {r'}_2^T + r'_2 {r'}_1^T \right) \left(r'_1 {r'}_2^T - r'_2 {r'}_1^T \right) \right] [I_1 - I_2]$$

$$= \frac{I_1 + I_2}{2} + \frac{1}{2} \frac{1}{1 - [{r'}_1^T r'_2]^2} \left[r'_1 [{r'}_2^T r'_1] {r'}_2^T - r'_2 [{r'}_1^T r'_2] {r'}_1^T \right] [I_1 - I_2]$$

$$= \frac{I_1 + I_2}{2} + \frac{1}{2} \frac{[{r'}_1^T r'_2]}{1 - [{r'}_1^T r'_2]^2} \left(r'_1 {r'}_2^T - r'_2 {r'}_1^T \right) [I_1 - I_2]$$

Consider the cross product, and its cross product matrix form

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\begin{aligned}(a \times b)\times &= \begin{bmatrix} 0 & -[a_1 b_2 - a_2 b_1] & a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 & 0 & -[a_2 b_3 - a_3 b_2] \\ -[a_3 b_1 - a_1 b_3] & a_2 b_3 - a_3 b_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_2 b_1 - a_1 b_2 & a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 & 0 & a_3 b_2 - a_2 b_3 \\ a_1 b_3 - a_3 b_1 & a_2 b_3 - a_3 b_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a_1 b_1 & a_2 b_1 & a_3 b_1 \\ a_1 b_2 & a_2 b_2 & a_3 b_2 \\ a_1 b_3 & a_2 b_3 & a_3 b_3 \end{bmatrix} - \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \\ &= b a^T - a b^T\end{aligned}$$

Similarly, we have

$$\left(r'_1 {r'}_2^T - r'_2 {r'}_1^T\right) = -(r'_1 \times r'_2) \times$$

$$M = \frac{I_1 + I_2}{2} - \frac{1}{2} \frac{[{r'}_1^T r'_2]}{1 - [{r'}_1^T r'_2]^2} (r'_1 \times r'_2) \times [I_1 - I_2]$$

under-water model

The unit vector on the laser plane v_m

$$
\begin{aligned}
v_m &= c_1 \pi + c_2 n \\
\pi^T v_m &= \pi^T (c_1 \pi + c_2 n) = c_1 + c_2 (\pi^T n) = 0 \\
v_m^T v_m &= (c_1^2 + c_2^2) + 2 c_1 c_2 (\pi^T n) = 1
\end{aligned}
$$

Then we can have that

$$
\begin{aligned}
c_1 &= -c_2 (\pi^T n) \\
[1 + (\pi^T n)^2] c_2^2 - 2(\pi^T n)^2 c_2^2 &= 1
\end{aligned}
$$

Then we solve

$$
\begin{aligned}
c_2 &= \frac{-1}{\sqrt{1 - (\pi^T n)^2}} \\
c_1 &= \frac{(\pi^T n)}{\sqrt{1 - (\pi^T n)^2}} \\
v_m &= \pi [\frac{(\pi^T n)}{\sqrt{1 - (\pi^T n)^2}}] + n [\frac{-1}{\sqrt{1 - (\pi^T n)^2}}]
\end{aligned}
$$

expression of r, r'

Consider replace symbol of refraction:

- replace v_1 with r to represent unit direction vector (FROM)
- replace v_2 with r' to represent unit direction vector (TO)

Here r_θ is the unit vector
after rotating v_m around rotaion axis π for angle θ:

$$
\begin{aligned}
r_\theta &= R_\theta v_m \\
&= [\cos\theta I + \sin\theta(\pi\times) + (1-\cos\theta)\pi\pi^T]v_m \\
&= [\cos\theta I + \sin\theta(\pi\times) + (1-\cos\theta)\pi\pi^T]\Big(\pi[\frac{(\pi^T n)}{\sqrt{1-(\pi^T n)^2}}] + n[\frac{-1}{\sqrt{1-(\pi^T n)^2}}]\Big) \\
&= \cos\theta\Big(\pi[\frac{(\pi^T n)}{\sqrt{1-(\pi^T n)^2}}] + n[\frac{-1}{\sqrt{1-(\pi^T n)^2}}]\Big) \\
&+ \sin\theta(\pi\times n)[\frac{-1}{\sqrt{1-(\pi^T n)^2}}]
\end{aligned}
$$

After refraction, the formula of r'_θ is:

$$
r'_\theta = n\left[(\frac{n_1}{n_2})[-n^T r_\theta] - \sqrt{1-(\frac{n_1}{n_2})^2(1-[r_\theta^T n]^2)}\right] + r_\theta(\frac{n_1}{n_2})
$$

Here, the inner product, is a function of angle θ:

$$
[-n^T r_\theta] = \cos\theta\sqrt{1-(\pi^T n)^2}
$$

Thus

$$
\begin{aligned}
r'_\theta = n&\left[-(\frac{n_1}{n_2})[\cos\theta\frac{(\pi^T n)^2}{\sqrt{1-(\pi^T n)^2}}] - \sqrt{1-(\frac{n_1}{n_2})^2\Big(1-\cos^2\theta[1-(\pi^T n)^2]\Big)}\right] \\
&+ \sin\theta(\pi\times n)(\frac{n_1}{n_2})[\frac{-1}{\sqrt{1-(\pi^T n)^2}}] \\
&+ \cos\theta\,\pi(\frac{n_1}{n_2})[\frac{(\pi^T n)}{\sqrt{1-(\pi^T n)^2}}]
\end{aligned}
$$

find intersection I

set the interface plane

$$
n^T I = h
$$

Here it must on the ray, all start from laser point P:

$$
I_\theta = P + k_\theta r_\theta
$$

Solve k_θ, I_θ respectively

$$n^T P - k_\theta \cos\theta \sqrt{1-(\pi^T n)^2} = h$$

$$k_\theta = \frac{n^T P - h}{\cos\theta \sqrt{1-(\pi^T n)^2}}$$

$$I_\theta = P + [n^T P - h] \left[\pi [\frac{(\pi^T n)}{1-(\pi^T n)^2}] + n[\frac{-1}{1-(\pi^T n)^2}] \right] + [n^T P - h] \tan\theta (\pi \times n) [\frac{-1}{1-(\pi^T n)^2}]$$