Rodrigues' rotation formula

Reference: https://www.cnblogs.com/xpvincent/archive/2013/02/15/2912836.html

symbol table

SYMBOL	MEANING
v	vector original
v'	vector rotated
z	unit vector of rotation axis
θ	rotation angle
x, y	unit vectors orthogonal
x', y'	unit vectors rotated

analysis

Here we can write v, v' as:

$$\begin{aligned} v &= ax + by + cz \\ v' &= ax' + by' + cz \end{aligned}$$

Relationship between unit vectors x', y' and x, y:

$$\begin{aligned} x' &= \cos\theta x + \sin\theta y \\ y' &= -\sin\theta x + \cos\theta y \end{aligned}$$

Represent vector rotated v' with unit vectors x, y, z:

$$\begin{aligned} v' &= a(\cos\theta x + \sin\theta y) + b(-\sin\theta x + \cos\theta y) + cz \\ &= \cos\theta(ax + by) + \sin\theta(ay - bx) + cz \end{aligned}$$

Represent $(ax + by), (ay - bx), (cz)$, with v, z respectively:

$$\begin{aligned} cz &= (v \cdot z)z \\ ax + by &= v - cz = v - (v \cdot z)z \\ ay - bx &= z \times (ax + by) = z \times (ax + by + cz) = z \times v \end{aligned}$$

Replace $(ax+by), (ay-bx), (cz)$,
with $[v-(v\cdot z)z], [z\times v], (v\cdot z)z$ respectively:

$$\begin{aligned} v' &= \cos\theta[v-(v\cdot z)z] + \sin\theta[z\times v] + (v\cdot z)z \\ &= v + \sin\theta[z\times v] + (1-\cos\theta)[-v+(v\cdot z)z] \end{aligned}$$

Write cross product $\times$ as Antisymmetric Matrix$[A^T = -A]$:

$$\begin{aligned} z\times v &= \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = Av \\ -v+(v\cdot z)z &= -(ax+by) \\ &= z\times(ay-bx) \\ &= z\times(z\times v) \\ &= A^2 v \end{aligned}$$

formula

Eventually, replace $[z\times v], [-v+(v\cdot z)z]$ with A, v:

$$\begin{aligned} v' &= [v] + \sin\theta[Av] + (1-\cos\theta)[A^2 v] \\ &= [I + \sin\theta A + (1-\cos\theta)A^2]v \end{aligned}$$

That means if we know $\theta, z \Leftrightarrow \theta, A$, we can calculate vector rotated v'

rotation matrix

Here define rotation matrix R, we have $v' = Rv$:

$$R \equiv I + \sin\theta A + (1-\cos\theta)A^2$$

Consider A^2 and z:

$$\begin{aligned} A^2 &= -A^T A = -\begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix} \\ &= -\begin{bmatrix} z_2^2+z_3^2 & -z_1z_2 & -z_1z_3 \\ -z_1z_2 & z_1^2+z_3^2 & -z_2z_3 \\ -z_1z_3 & -z_2z_3 & z_1^2+z_2^2 \end{bmatrix} \\ &= \begin{bmatrix} z_1^2-1 & z_1z_2 & z_1z_3 \\ z_1z_2 & z_2^2-1 & -z_2z_3 \\ z_1z_3 & z_2z_3 & z_3^2-1 \end{bmatrix} \\ &= zz^T - I \end{aligned}$$

Replace A^2 with $zz^T - I$:

$$\begin{aligned}R &\equiv I + \sin\theta A + (1-\cos\theta)[zz^T - I] \\ &= \cos\theta I + \sin\theta A + (1-\cos\theta)zz^T\end{aligned}$$

Euler formula

notice $Az = z \times z = \vec{0}, z^T z = 1$,
and

$$\begin{aligned}(zz^T - I)^k &= \sum_{i=0}^{k}\binom{k}{i}(zz^T)^i(-1)^{k-i}I^{k-i} \\ &= (-1)^k[I + (zz^T)\sum_{i=1}^{k}\binom{k}{i}(-1)^i] \quad k \geq 1 \\ &= (-1)^k[I + (zz^T)[\sum_{i=0}^{k}\binom{k}{i}(-1)^i - 1]] \quad k \geq 1 \\ &= (-1)^k[I - zz^T] \qquad k \geq 1 \\ (zz^T - I)^k &= (-1)^k I \quad k = 0\end{aligned}$$

So, with these

$$\begin{aligned}\exp(\theta A) &= \sum_{n=0}^{\infty}\frac{1}{n!}(\theta A)^n \\ &= \sum_{k=0}^{\infty}\frac{1}{(2k)!}(\theta A)^{2k} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}(\theta A)^{2k+1} \\ &= \sum_{k=0}^{\infty}\frac{1}{(2k)!}\theta^{2k}(zz^T - I)^k + A\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\theta^{2k+1}(zz^T - I)^k \\ &= zz^T + \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\theta^{2k}[I - zz^T] + A\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\theta^{2k+1}I \\ &= zz^T + \cos\theta[I - zz^T] + \sin\theta A \\ &= \cos\theta I + \sin\theta A + (1-\cos\theta)zz^T\end{aligned}$$

To sum up

$$\begin{aligned}R &\equiv I + \sin\theta A + (1-\cos\theta)A^2 \\ &= I + \sin\theta A + (1-\cos\theta)[zz^T - I] \\ &= \cos\theta I + \sin\theta A + (1-\cos\theta)zz^T \\ &= \exp(\theta A)\end{aligned}$$

quaterion

Think about the quaterion q:

$$q = s + u = \begin{bmatrix} s \\ u \end{bmatrix} = \begin{bmatrix} s \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = s\ 1 + (u_1\ i + u_2\ j + u_3\ k)$$

Rules of quaterion computation are

$$\begin{aligned} ijk = i^2 = j^2 = k^2 = -1 \\ ij = k, jk = i, ki = j \end{aligned}$$

Product of any q_1, q_2, notice here u_1, u_2 are vectors, s_1, s_2 are scalars and notice $u_1 u_2 = -(u_1^T u_2) + u_1 \times u_2$:

$$\begin{aligned} q_1 q_2 &= (s_1 + u_1)(s_2 + u_2) = (s_1 s_2 + u_1 u_2) + s_1 u_1 + s_2 u_2 \\ &= (s_1 s_2 - u_1^T u_2) + [s_1 u_2 + s_2 u_1] + u_1 \times u_2 \\ &= \begin{bmatrix} s_1 s_2 - u_1^T u_2 \\ [s_1 u_2 + s_2 u_1] + u_1 \times u_2 \end{bmatrix} \end{aligned}$$

Then think about the quaterion of vector $p = [0; v]$,
and normal quaterion $q \quad (|q|^2 = s^2 + |u|^2 = 1)$,
notice that $-(u \times v) \times u = u \times (u \times v) = -[v(u^T u) - u(u^T v)]$:

$$\begin{aligned} q' \equiv qpq* &= \begin{bmatrix} s \\ u \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} \begin{bmatrix} s \\ -u \end{bmatrix} \\ &= \begin{bmatrix} -u^T v \\ sv + u \times v \end{bmatrix} \begin{bmatrix} s \\ -u \end{bmatrix} \\ &= \begin{bmatrix} -su^T v + su^T v \\ uu^T v + s^2 v + su \times v - sv \times u - u \times v \times u \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ v(s^2 + uu^T) + 2su \times v - (u \times v) \times u \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ v(s^2 + uu^T) + 2su \times v - v(u^T u) + u(u^T v) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ v(s^2 - u^T u + 2uu^T) + 2su \times v \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ [(s^2 - u^T u)I + 2su \times + 2uu^T]v \end{bmatrix} \end{aligned}$$

Compare with rotation matrix R:

$$\begin{aligned} R &\equiv \cos\theta I + \sin\theta A + (1 - \cos\theta) zz^T \\ &= \cos\theta I + \sin\theta (z\times) + (1 - \cos\theta) zz^T \end{aligned}$$

Try to make

$$
\begin{aligned}
|q|^2 &= s^2 + u^T u = 1 \\
(s^2 - u^T u) &= \cos\theta \\
2s\,u &= \sin\theta\,z \\
2uu^T &= (1-\cos\theta)zz^T
\end{aligned}
$$

Set parameter λ, to make $u = \lambda z$, notice $z^T z = 1$ then:

$$
\begin{aligned}
s^2 &= 1 - \lambda^2 \\
(1 - 2\lambda^2) &= \cos\theta \\
2 \pm \sqrt{1-\lambda^2}\lambda &= \sin\theta \\
2\lambda^2 &= 1 - \cos\theta
\end{aligned}
$$

So when we select $\lambda = \sin\frac{\theta}{2}$, notice z is unit vector of rotation axis, must have:

$$
\begin{aligned}
s &= \cos\frac{\theta}{2} \\
u &= (\sin\frac{\theta}{2})z \\
q = \begin{bmatrix} s \\ u \end{bmatrix} &= \begin{bmatrix} \cos\frac{\theta}{2} \\ (\sin\frac{\theta}{2})z \end{bmatrix}
\end{aligned}
$$

quaterion can implement rotation, z is unit vector of rotation axis, θ is rotation angle:

$$
\begin{aligned}
p' \equiv qpq* &= \begin{bmatrix} s \\ u \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} \begin{bmatrix} s \\ -u \end{bmatrix} \\
&= \begin{bmatrix} \cos\frac{\theta}{2} \\ (\sin\frac{\theta}{2})z \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ -(\sin\frac{\theta}{2})z \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ [(s^2 - u^T u)I + 2su \times + 2uu^T]v \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ [\cos\theta I + \sin\theta(z\times) + (1-\cos\theta)zz^T]v \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ [\cos\theta I + \sin\theta A + (1-\cos\theta)zz^T]v \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ v' \end{bmatrix}
\end{aligned}
$$

Euler formula of quaterion

Consider the Euler formula of quaterion,

notice that $\begin{bmatrix} 0 \\ z \end{bmatrix}^2 = \begin{bmatrix} 0 - z^T z \\ 0z + 0z + z \times z \end{bmatrix} = -1$:

$$
\begin{aligned}
\exp\left(\begin{bmatrix} 0 \\ (\frac{\theta}{2})z \end{bmatrix}\right) &\equiv \sum_{n=0}^{\infty} \frac{(\frac{\theta}{2})^n}{n!} \begin{bmatrix} 0 \\ z \end{bmatrix}^n \\
&= \sum_{k=0}^{\infty} \frac{(\frac{\theta}{2})^{2k}}{(2k)!}(-1)^k + \begin{bmatrix} 0 \\ z \end{bmatrix} \sum_{k=0}^{\infty} \frac{(\frac{\theta}{2})^{2k+1}}{(2k)!}(-1)^k \\
&= \cos\frac{\theta}{2} + \begin{bmatrix} 0 \\ z \end{bmatrix} \sin\frac{\theta}{2} \\
&= \begin{bmatrix} \cos\frac{\theta}{2} \\ (\sin\frac{\theta}{2})z \end{bmatrix} \\
&= q
\end{aligned}
$$