Bonus.

If it takes 3 hours to roast a 15 lb turkey, how long will it take to roast a 20 lb one in the same oven? Give your best estimate and explain your method.

solution

Set symbols as follow:
$k \equiv \frac{C\rho}{K}$, where C is Specific heat capacity, ρ is density, K is the heat conductivity
$u(r,t)$ is temperature of turkey, u_e is the temperature of oven, u_0 is the initial temperature of turkey, here we treat the the turkey as a sphere of radius R
h is the Heat transfer coefficient of Newton cooling

Based on the Example 6.13 in the textbook, we can write equation

$$\frac{\partial u}{\partial t} = k\Delta u$$

$$u(r,t)|_{t=0} = u_0, \quad -K\frac{\partial u}{\partial r}|_{r=R} = h(u-u_e)|_{r=R}, \quad \frac{\partial u}{\partial r}|_{r=0} = 0$$

Now, define $v = u - u_e$, and the boundary conditions are symmetric for angle angles, thus $v = v(r,t)$

$$\Delta(\cdot) = \left[\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right]\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right](\cdot)$$

Thus

$$\frac{\partial v}{\partial t} = k\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left[r^2\frac{\partial v}{\partial r}\right]\right)$$

$$v(r,t)|_{t=0} = u_0 - u_e, \quad \left(hv + K\frac{\partial v}{\partial r}\right)|_{r=R} = 0, \quad \frac{\partial v}{\partial r}|_{r=0} = 0$$

Consider to write the function v as the weighted sum of separable $v_n = y_n(r)\phi_n(t)$

$$v(r,t) = \sum c_n y_n(r)\phi_n(t) \Rightarrow \frac{-\frac{1}{r^2}\left(\frac{d}{dr}\left[r^2\frac{dy_n(r)}{dr}\right]\right)}{y_n(r)} = \frac{-\frac{d\phi_n(t)}{dt}}{k\phi_n(t)} = \bar{\lambda}_n$$

It leads to

$$-\frac{1}{r^2}\left(\frac{d}{dr}\left[r^2\frac{dy_n}{dr}\right]\right) = \bar{\lambda}_n y_n$$

$$-\frac{d\phi_n(t)}{dt} = k\bar{\lambda}_n\phi_n(t)$$

Here we can solve $\phi_n(t)$

$$\phi_n(t) = e^{-k\bar{\lambda}_n t}$$

Let's come back to the y_n

$$-\left(\frac{d}{dr}\left[r^2\frac{dy_n}{dr}\right]\right) = \bar{\lambda}_n r^2 y_n$$

$$\left(hy_n + K\frac{dy_n}{dr}\right)|_{r=R} = 0, \quad \frac{dy_n}{dr}|_{r=0} = 0$$

To eliminate R, assume **heat transfer$\gg$heat diffusion**, $h \gg \frac{K}{R}$ at $r = R$. Substitute $x \equiv r/R$,

$$-\left(\frac{d}{dx}\left[x^2\frac{dy_n}{dx}\right]\right) = \bar{\lambda}_n R^2 x^2 y_n = \lambda_n x^2 y_n$$

$$y_n|_{x=1} = 0, \quad \frac{dy_n}{dx}|_{x=0} = 0$$

So, we have the the correspondence, here λ_n is a fixed value for any R

$$\bar{\lambda}_n = \frac{\lambda_n}{R^2}, \quad \phi_n(t) = e^{-k\lambda_n \frac{t}{R^2}}$$

The equation of y_n is a SL problem, here λ_n is the eigenvalue, y_n is the corresponding eigenfunction

$$\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] + q(x), \quad p(x) = x^2, \quad q(x) = 0$$

$$\mathcal{L}y_n = \lambda_n w(x) y_n, \quad w(x) = x^2$$

We can verify that

$$\int_a^b y_n \frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right] dx = \left[y_n p(x)\frac{dy_m}{dx}\right]_a^b - \int_a^b p(x)\frac{dy_n}{dx}\frac{dy_m}{dx} dx \textbf{ symmetric form}$$

$$\int_a^b y_n q(x) y_m dx \textbf{ symmetric form}$$

Thus

$$\int_a^b y_n \mathcal{L} y_m - y_m \mathcal{L} y_n = \left[y_m p(x)\frac{dy_n}{dx} - y_n p(x)\frac{dy_m}{dx}\right]_a^b = \left[p(x)\left(y_m\frac{dy_n}{dx} - y_n\frac{dy_m}{dx}\right)\right]_a^b$$

Set $[a,b] = [0,1]$, notice $p(0) = 0$, $\frac{dy_n}{dx}|_{x=0} = 0$, $\frac{dy_m}{dx}|_{x=0} = 0$

$$\left[p(x)\left(y_m\frac{dy_n}{dx} - y_n\frac{dy_m}{dx}\right)\right]_0^1 = 0 - 0 = 0$$

$$\int_0^1 y_n \mathcal{L} y_m - y_m \mathcal{L} y_n = \int_0^1 y_n \lambda_m w(x) y_m - y_m \lambda_n w(x) y_n = 0$$

We can define the bracket as

$$\langle f, g\rangle = \int_0^1 f\ w(x)\ g dx, \quad \langle y_n, \lambda_m y_m\rangle = \int_0^1 y_n w(x)\lambda_m y_m dx, \quad \langle \lambda_n y_n, y_m\rangle = \int_0^1 \lambda_n y_n w(x) y_m dx$$

$$\langle \lambda_n y_n, y_m\rangle = \langle y_n, \lambda_m y_m\rangle$$

For $\lambda_n \neq \lambda_m$

$$\langle y_n, y_m\rangle = 0 \Leftrightarrow (\lambda_n - \lambda_m)\langle y_n, y_m\rangle = 0 \Leftrightarrow \langle \lambda_n y_n, y_m\rangle = \langle y_n, \lambda_m y_m\rangle$$

If λ_n has multiple eigenfunctions y_n, y_n', Gram–Schmidt process can make sure $\langle y_n, y_n'\rangle = 0$
Currently, we can calculate c_n, with the boundary condition

$$v(r,t)|_{t=0} = \sum c_n y_n\left(\frac{r}{R}\right)\phi_n(t)|_{t=0} = \sum c_n y_n(x) e^{-k\lambda_n \frac{t}{R^2}}|_{t=0} = \sum c_n y_n(x) = u_0 - u_e$$

$$(u_0 - u_e)\langle 1, y_n\rangle = \langle u_0 - u_e, y_n\rangle = \langle \sum c_n y_n, y_n\rangle = c_n\langle y_n, y_n\rangle$$

Thus

$$c_n = (u_0 - u_e)\frac{\langle 1, y_n\rangle}{\langle y_n, y_n\rangle} = (u_0 - u_e)\frac{\int_0^1 1\ w(x) y_n dx}{\int_0^1 y_n w(x) y_n dx} = (u_0 - u_e)\frac{\int_0^1 x^2 y_n dx}{\int_0^1 x^2 y_n^2 dx}$$

$$u(r,t) = v(r,t) + u_e = (u_0 - u_e)\left[\sum \frac{\int_0^1 x^2 y_n dx}{\int_0^1 x^2 y_n^2 dx} y_n\left(\frac{r}{R}\right) e^{-k\lambda_n \frac{t}{R^2}}\right] + u_e$$

Expand the eigenfunction $y(x) = \sum_{k=0}^{\infty} a_k x^k$, compare the coefficient of x^k

$$-(k+1)ka_k = \lambda a_{k-2} \Rightarrow \frac{a_k}{a_{k-2}} = \frac{(-\lambda)}{(k+1)k}, \quad \frac{dy}{dx}|_{x=0} = a_1 = 0 \Rightarrow a_{2k+1} = 0, \ k \in \mathbb{Z}^*$$

notice that, set $y(0) = a_0 = 1$

$$\frac{a_{2k}}{a_0} = \prod_{l=1}^{k} \frac{a_{2l}}{a_{2l-2}} = \prod_{l=1}^{k} \frac{(-\lambda)}{(l+1)l} = \frac{(-\lambda)^k}{(2k+1)!} \Rightarrow a_{2k} = \frac{(-\lambda)^k}{(2k+1)!}$$

With the boundary condition for eigenfunction $y(1) = 0$

$$y(1) = \sum_{k=0}^{\infty} a_{2k} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(2k+1)!} = \frac{1}{\sqrt{\lambda}} \sum_{k=0}^{\infty} (-1)^k \frac{(\sqrt{\lambda})^{2k+1}}{(2k+1)!} = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = 0$$

Thus

$$\sqrt{\lambda_n} = n\pi \Rightarrow \lambda_n = n^2\pi^2, \ y_n(x) = \sum_{k=0}^{\infty} \frac{(-n^2\pi^2)^k}{(2k+1)!} x^{2k} = \frac{\sin(n\pi x)}{n\pi x}, \quad n \in \mathbb{Z}^+$$

For the coefficientc_n

$$c_n = \frac{\int_0^1 x^2 y_n dx}{\int_0^1 x^2 y_n^2 dx} = \frac{\int_0^1 x^2 \frac{\sin(n\pi x)}{n\pi x} dx}{\int_0^1 x^2 \frac{\sin^2(n\pi x)}{n^2\pi^2 x^2} dx} = \frac{\frac{(-1)^{n+1}}{n^2\pi^2}}{\frac{1}{2n^2\pi^2}} = 2(-1)^{n+1}$$

$$u(r,t) = (u_0 - u_e) \left[\sum_{n=1}^{\infty} 2(-1)^{n+1} \frac{\sin(n\pi \frac{r}{R})}{n\pi \frac{r}{R}} e^{-kn^2\pi^2 \frac{t}{R^2}} \right] + u_e$$

The meaning of the cooked turkey is that the inner temperature must reach the temperature threshold u_{th} at the cooked time t^*. i.e. $u(r,t)|_{r=0,t=t^*} = u_{th}$

$$u(r,t)|_{r=0,t=t^*} = (u_0 - u_e) \left[\sum_{n=1}^{\infty} 2(-1)^{n+1} e^{-kn^2\pi^2 \frac{t^*}{R^2}} \right] + u_e = u_{th}$$

notice that $\lambda_n, y_n, k, u_e, u_0$ won't change when the radius R of turkey changes, it implies

$$\frac{t^*}{R^2} = f\left(\frac{u_{th} - u_e}{u_0 - u_e}\right) / k = \text{const}$$

That is the relationship of cooking time t^* and radius R,
under the assumption: **heat transfer** $\gg$ **heat diffusion**, i.e. i.e. $h \gg \frac{K}{R}$ at $r = R$

$$\frac{R^3}{m} = \frac{3}{4\pi\rho} = \text{const}$$

Finally

$$\frac{t^*}{m^{\frac{2}{3}}} = \frac{t^*}{R^2} \cdot \left[\frac{R^3}{m}\right]^{\frac{2}{3}} = \text{const}$$

It takes t_1^*=3 hours to roast m_1=15 lb turkey, we wan to know how long will it take t_2^* to roast a m_2 =20 lb one in the same oven

$$\frac{t_1^*}{m_1^{\frac{2}{3}}} = \frac{t_2^*}{m_2^{\frac{2}{3}}} = \text{const} \Longrightarrow t_2^* = \left(\frac{m_2}{m_1}\right)^{\frac{2}{3}} t_1^* = \left(\frac{20}{15}\right)^{\frac{2}{3}} \cdot 3 = 3.634241 \approx 3.63$$

To sum up, it take $t_2^* \approx 3.63$ hours to roast a $m_2 = 20$ lb one in the same oven

Compare and contrast the eigenvalue problems for square matrices and for differential operators. In particular, discuss the difference and similarity regarding the number of eigenvalues and the meaning of orthogonality.

solution
Differential operator $\mathcal{L}$
Think about the general equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y + \alpha(x)\frac{\mathrm{d}}{\mathrm{d}x}y + \beta(x)y = f(x)$$

We can all convert into this form

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)y = \lambda w(x)f(x)$$

We want to expand $y,\ f$ in such form

$$y = \sum_{n=0}^{\infty} c_n y_n,\ \ f(x) = \sum_{n=0}^{\infty} d_n y_n$$

To calculate c_n, assume: for eigenvalue $\lambda_n = d_n/c_n$, we have eigenfunction y_n
For the SL problem, here $x \in [a,b]$, the operator $\mathcal{L}$

$$\mathcal{L}y_n = -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right] + q(x)y_n = \lambda_n w(x)y_n$$

$$\mathcal{L}y = \sum_{n=0}^{\infty} c_n\lambda_n w(x)y_n = w(x)\sum_{n=0}^{\infty} d_n y_n = w(x)f(x)$$

Now, if λ_n, y_n are known, we want to find the expression of d_n, then we can conclude $c_n = d_n/\lambda_n$
Guess there is a bracket operator $\langle\cdot,\cdot\rangle$

$$\langle f, y_n\rangle = d_n\langle y_n, y_n\rangle + \sum_{m\neq n} d_m\langle y_m, y_n\rangle$$

If $\langle y_m, y_n\rangle = 0$ for $m \neq n$, we can conclude $d_n = \langle f, y_n\rangle/\langle y_n, y_n\rangle$
So, what the bracket operator $\langle\cdot,\cdot\rangle$ should be, let's guess, for $\lambda_n \neq \lambda_m$

$$\langle y_n, y_m\rangle = 0 \Leftrightarrow (\lambda_n - \lambda_m)\langle y_n, y_m\rangle = 0 \Leftrightarrow \langle \lambda_n y_n, y_m\rangle = \langle y_n, \lambda_m y_m\rangle$$

think about the relationship $\lambda_n w(x)y_n = \mathcal{L}y_n$ notice that formula

$$\int_a^b y_n\frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right]dx = \left[y_n p(x)\frac{dy_m}{dx}\right]_a^b - \int_a^b p(x)\frac{dy_n}{dx}\frac{dy_m}{dx}dx \ \textbf{symmetric form}$$

$$\int_a^b y_n q(x) y_m dx \ \textbf{symmetric form}$$

here

$$\int_a^b y_n\mathcal{L}y_m - y_m\mathcal{L}y_n = \left[y_m p(x)\frac{dy_n}{dx} - y_n p(x)\frac{dy_m}{dx}\right]_a^b = \left[p(x)\left(y_m\frac{dy_n}{dx} - y_n\frac{dy_m}{dx}\right)\right]_a^b$$

If we have (i) $p(a) = p(b) = 0$, (ii) $p(a) = p(b), y_n(a) = y_n(b), y_n'(a) = y_n'(b)$,
(iii) $\alpha_1 y_n(a) + \alpha_2 y_n'(a) = 0, \beta_1 y_n(b) + \beta_2 y_n'(b) = 0$ for all n

$$\int_a^b y_n \mathcal{L} y_m - y_m \mathcal{L} y_n = \int_a^b y_n \lambda_m w(x) y_m - y_m \lambda_n w(x) y_n = 0$$

now we can define the bracket as

$$\langle u, v\rangle = \int_a^b u\ w(x)\ v dx, \quad \langle y_n, \lambda_m y_m\rangle = \int_a^b y_n w(x) \lambda_m y_m dx, \quad \langle \lambda_n y_n, y_m\rangle = \int_a^b \lambda_n y_n w(x) y_m dx$$

$$\langle \lambda_n y_n, y_m\rangle = \langle y_n, \lambda_m y_m\rangle$$

If λ_n has multiple eigenfunctions y_m, y_n', **Gram–Schmidt process** can make sure $\langle y_n, y_n'\rangle = 0$

Square matrice A
For square matrix A

$$AX = b$$

we can first expand $X = \sum c_n X_n, b = \sum d_n X_n$, then assume we have independent equations

$$c_n A X_n = d_n X_n \Leftrightarrow A X_n = (d_n/c_n) X_n = \lambda_n X_n$$

once we know d_n, λ_n, we can determine $c_n = d_n/\lambda_n$ directly
especially, when $A^T = A$, for $\lambda_n \neq \lambda_m$

$$\lambda_n X_m^T X_n = X_m^T A X_n = X_m^T A^T X_n = \lambda_m X_m^T X_n \Leftrightarrow X_m^T X_n = 0$$

here we obtain d_n with

$$X_n^T b = X_n^T \sum d_n X_n = d_n X_n^T X_n \Rightarrow d_n = X_n^T b / X_n^T X_n$$

If λ_n has multiple eigenvectors X_n, X_n', **Gram–Schmidt process** can make sure $X_n^T X_n' = 0$

Similarity
The operator $\mathcal{L}$, square matrix A, they both use the orthogonality:$\langle y_m, y_n\rangle = 0, X_m^T X_n = 0$ $(m \neq n)$
to determine the coefficients c_n of components: eigenfunction y_n, eigenvector X_n

Difference
For the operator $\mathcal{L}$, it **could** have **infinite countable** eigenvalues
especially, when it satisfy (iii) $\alpha_1 y_n(a) + \alpha_2 y_n'(a) = 0, \beta_1 y_n(b) + \beta_2 y_n'(b) = 0$ for all n,
each λ_n only has one eigenfunction y_n,
otherwise, with eigenfunctions y_n, y_n', (iii)$\Rightarrow W(a) = 0 \Rightarrow W(x) = 0 \Rightarrow y_n, y_n'$ linearly dependent
(note: λ_n could have multiple eigenfunctions f_n (e.g. section 5.2.2 Problem 3)
when it satisfies (i) $p(a) = p(b) = 0$ or (ii) $p(a) = p(b), y_n(a) = y_n(b), y_n'(a) = y_n'(b)$ instead of (iii))

For the square matrix $A = A^T$, it only has **finite** eigenvalues,
each λ_n could have multiple eigenfunction X_n