Chap 1.1 Dimension Analysis

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Agenda

- Fundamental dimensions
- Generalized Π theorem
- Process
  - Represent quantities $q_1, \cdots, q_m$
  - Express dimensionless π
  - Write dimension matrix A
  - Row operation $A = [R_{n\times r} | T_{n\times(m-r)}] \xrightarrow{\text{row operation } S} SA = \begin{bmatrix} I_{r\times r} & | & Q \\ 0 & & 0 \end{bmatrix}$
  - Solve $A\mathrm{p} = 0$
  - Physical law with π
  - Simplify

Fundamental dimensions

Table: Fundamental dimensions

Dimension	Unit	Symbol	Unit Abbr.
Mass	kilogram	M	kg
Length	meter	L	m
Time	second	T	s
Electric Current	ampere	I	A
Luminous Intensity	candela	J	cd
Temperature	kelvin	Θ	K
Amount of Substance	mole	N	mol

Generalized Π theorem

A physical law relating m dimensional quantities $q_1, ..., q_m$

$$f(q_1, ..., q_m) = 0$$

is equivalent to

$$F(\pi_1, ..., \pi_r) = 0$$

that relates the r dimensionless quantities $\pi_1, ..., \pi_r$ that can be formed from $q_1, ..., q_m$

Process

- Represent quantities

Example 1.7

- (Heat transfer) $f(e, t, r, c, k, u) = 0$ with quantities e, t, r, c, k, u
- $[e] = ML^2T^{-2}, [t] = T, [r] = L, [c] = ML^{-1}T^{-2}\Theta^{-1}$,
  $[k] = L^2T^{-1}, [u] = \Theta$

Process

- Express dimensionless π

Example 1.7

- $\pi = (e)^{q_1}(t)^{q_2}(r)^{q_3}(c)^{q_4}(k)^{q_5}(u)^{q_6}$
- $[\pi] = (ML^2T^{-2})^{q_1}(T)^{q_2}(L)^{q_3}(ML^{-1}T^{-2}\Theta^{-1})^{q_4}(L^2T^{-1})^{q_5}(\Theta)^{q_6}$
-

	e	t	r	c	k	u
M	1	0	0	1	0	0
L	2	0	1	-1	2	0
T	-2	1	0	-2	-1	0
Θ	0	0	0	-1	0	1

Process

- Write dimension matrix A

Example 1.7

	e	t	r	c	k	u
M	1	0	0	1	0	0
L	2	0	1	-1	2	0
T	-2	1	0	-2	-1	0
Θ	0	0	0	-1	0	1

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & -1 & 2 & 0 \\ -2 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

Process

- Row operation $A_{n\times m} \quad (n < m)$
  - $r \equiv \text{rank}(A) = n$

    $$A_{n\times m} = [R_{n\times n}|T_{n\times(m-n)}] \xrightarrow{\text{row operation } R^{-1}} R^{-1}A = [I_{n\times n}|R^{-1}T]$$
  - $r \equiv \text{rank}(A) < n$

    $$A_{n\times m} = [R_{n\times r}|T_{n\times(m-r)}] \xrightarrow{\text{row operation } S} SA = \left[\begin{array}{c|c} I_{r\times r} & Q \\ 0 & 0 \end{array}\right]$$
  - note

    select columns of quantities wisely to make: $\text{rank}(R) = r$

    always holds $\text{rank}(S) \equiv n$

Example 1.7

- $$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & -1 & 2 & 0 \\ -2 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \quad SA = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

Process

- Solve $A\mathrm{p} = 0$
  - $A\mathrm{p} = 0 \Leftrightarrow (SA)\mathrm{p} = \left[\begin{smallmatrix} I_{r\times r} & \vert & Q \\ 0 & & 0 \end{smallmatrix}\right]\mathrm{p} = 0 \Leftrightarrow [I_{r\times r}|Q]\mathrm{p} = 0$
  - $[I_{r\times r}|Q] \cdot \left[\begin{smallmatrix} -Q_i \\ E_i \end{smallmatrix}\right] = 0, \quad E_i \equiv \underbrace{[\overbrace{0\cdots 0}^{(i-1)\times 0} 1 0\cdots 0]^T}_{n-r}, Q_i \equiv QE_i$ i-th col. of Q
  - $\mathrm{p} = \sum_{i=1}^{n-r} k_i \left[\begin{smallmatrix} -Q_i \\ E_i \end{smallmatrix}\right] \quad \forall k_i \in \mathbb{R}$

Example 1.7

- $$\mathrm{p} = k_1 \begin{bmatrix} -\begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix} + k_2 \begin{bmatrix} -\begin{pmatrix} 1 \\ 0 \\ -3 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Process

- Physical law with π
  - $F(\pi_1, ..., \pi_r) = 0$ with r dimensionless quantities $\pi_1, ..., \pi_r$

Example 1.7

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	e	t	r	c	k	u
π_1	0	1	-2	0	1	0
π_2	-1	0	3	1	0	1

$\pi_1 = \frac{tk}{r^2}, \pi_2 = \frac{cr^3u}{e}$

- $F(\pi_1, \pi_2) = F(\frac{tk}{r^2}, \frac{cr^3u}{e}) = 0$
- put what we want to know $u, ...$ in last columns, appear once in π_i

Process

- Simplify

Example 1.7

- $\pi_2 = \frac{cr^3 u}{e} = g(\pi_1) = g(\frac{tk}{r^2})$
- $u = \frac{e}{cr^3} g(\frac{tk}{r^2})$