## Bonus

(Age of the Earth) In this problem we use Lord Kelvin's argument, given in the mid 1860s, to estimate the age of the earth using a measurement of the geothermal gradient. The geothermal gradient is the derivative of temperature $T_x(0,t)$ measured at the surface of the earth. Since the earth is cooling,the temperature gradient is also decreasing with time. Lord Kelvin's idea of estimating the age of the earth $t^*$ is to treat the earth as flat with $x > 0$ measuring the depth from the surface $x = 0$; assume that the temperature on the surface of the earth was always $0°C$; and solve the heat conduction problem for the temperature of the earth

| | | |
|---|---|---|
| PDE | $C\rho T_t - KT_{xx} = 0$ | for $x > 0, t > 0$ |
| BC | $T(0,t) = 0$ | for $t > 0$ |
| IC | $T(x,0) = T_0$ | for $x > 0$ |

After finding the solution, we can find the time $t^*$ at which $T_x(0,t)|_{t=t^*}$ equals the current geothermal gradient value of $0.037°C/m$

(a) Solve the initial-boundary value problem.
(b) Assume that $K/C\rho \approx 1.2 \times 10^{-6} m^2/s$ and the initial temperature of the earth was $T_0 \approx 2000°C$(molten rock). Estimate the age of the earth $t^*$ by solving $T_x(x,t)|_{x=0,t=t^*} = 0.037°C/m$. Give your answer in millions of years.

**solution**

Do the Laplace transform for variable $t$

$$\mathcal{L}[T(x,t)] = U(x,s),\ \mathcal{L}[\frac{\partial T}{\partial t}] = sU(x,s) - T(x,t)|_{t=0} = sU(x,s) - T_0,$$
$$\mathcal{L}[\frac{\partial^2 T}{\partial x^2}] = \frac{\partial^2 U(x,s)}{\partial x^2},\ \mathcal{L}[T(x,t)|_{x=0}] = U(x,s)|_{x=0} = 0$$

Define $D = \frac{K}{C\rho}$, and it leads to

$$\frac{\partial^2 U(x,s)}{\partial x^2} = \frac{s}{D}U(x,s) - \frac{T_0}{D},\ U(x,s)|_{x=0} = 0$$

We can write $U(x,s) = U_h + U_p$, where the particular solution $U_p = \frac{T_0}{s}$, the homogeneous solution $U_h$ is given by

$$U_h = c_1(s)e^{\sqrt{\frac{s}{D}}x} + c_2(s)e^{-\sqrt{\frac{s}{D}}x}$$

Since $T(x,t) < M$ are bounded, $U = U_h + U_p < \frac{M}{s}$ still holds when $x \to \infty$, thus $c_1(s) = 0$

$$U(x,s) = U_p + U_h = \frac{T_0}{s} + c_2(s)e^{-\sqrt{\frac{s}{D}}x}$$

Compare with the condition

$$U(x,s)|_{x=0} = \frac{T_0}{s} + c_2(s)\cdot 1 = 0 \Rightarrow c_2(s) = -\frac{T_0}{s}$$

Thus, we have

$$U(x,s) = T_0\left[\frac{1}{s} - \frac{e^{-\sqrt{\frac{s}{D}}x}}{s}\right]$$

Then look up table of Laplace transform

$$\mathcal{L}^{-1}[\frac{e^{-a'\sqrt{s}}}{s}] = 1 - \text{erf}\left(\frac{a'}{2\sqrt{t}}\right) \quad ,a' = \frac{x}{\sqrt{D}}, \text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-z'^2}dz'$$

Thus

$$T(x,t) = \mathcal{L}^{-1}[U(x,s)] = T_0 \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right) = T_0 \cdot \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-z'^2} dz'$$

For the gradient of $x$ at $x = 0, t = t^*$

$$T_x(x,t)|_{x=0,t=t^*} = T_0 \cdot \frac{2}{\sqrt{\pi}} e^{-\left(\frac{x}{2\sqrt{Dt}}\right)^2} \cdot \frac{1}{2\sqrt{Dt}}|_{x=0,t=t^*} = T_0 \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{Dt^*}} = \frac{T_0}{\sqrt{\pi D t^*}} = 0.037°C/m$$

In the end, the age of earth $t^*$ is

$$t^* = \frac{T_0^2}{0.037^2 \cdot \pi D} \approx \frac{2000^2}{0.037^2 \pi \times 1.2 \times 10^6} \approx 7.750423 \times 10^{14} (s) = 24576430 \text{ (year)} \approx 25 \text{(milion year)}$$