## Problem 5

5. A radially symmetric solution to Laplace's equation $\Delta u = 0$ in $\mathbb{R}^3$ is a solution of the form

$$u = u(r), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

where $u$ depends only on the distance from the origin. Find all radially symmetric solutions in $\mathbb{R}^3$ (Section 6.3, Problems 5)

**solution**

The relationship between Cartesian coordinate system & Spherical coordinate system

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}$$

Laplace operator $\Delta$ in Cartesian coordinate system

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

**step 1** to express $\Delta$ with derivative operators regarding $r, \theta, \varphi$

$$\begin{aligned} \tfrac{\partial}{\partial r} &= [\sin\theta\cos\varphi \;\; \sin\theta\sin\varphi \;\; \cos\theta] \cdot [\tfrac{\partial}{\partial x} \;\; \tfrac{\partial}{\partial y} \;\; \tfrac{\partial}{\partial z}]^T \\ \tfrac{\partial}{\partial \theta} &= [r\cos\theta\cos\varphi \;\; r\cos\theta\sin\varphi \;\; -r\sin\theta] \cdot [\tfrac{\partial}{\partial x} \;\; \tfrac{\partial}{\partial y} \;\; \tfrac{\partial}{\partial z}]^T \\ \tfrac{\partial}{\partial \varphi} &= [-r\sin\theta\sin\varphi \;\; r\sin\theta\cos\varphi \;\; 0] \cdot [\tfrac{\partial}{\partial x} \;\; \tfrac{\partial}{\partial y} \;\; \tfrac{\partial}{\partial z}]^T \end{aligned}$$

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

define $A$, moreover, we can verify that $A^T \cdot A = I \Rightarrow A^{-1} = A^T$

$$A \equiv \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} = [A_1 \; A_2 \; A_3], \quad \partial = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = A^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \end{pmatrix} = A^T \partial_\mu$$

use simple signs to indicate $\partial \equiv [\frac{\partial}{\partial x} \;\; \frac{\partial}{\partial y} \;\; \frac{\partial}{\partial z}]^T$, and $\partial_\mu \equiv [\frac{\partial}{\partial r} \quad \frac{1}{r}\frac{\partial}{\partial \theta} \quad \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}]^T$

$$\Delta(\cdot) = \sum_{k=1}^{3} A_k^T \partial_\mu \left(A_k^T \partial_\mu(\cdot)\right) = \sum_{k=1}^{3} A_k^T \left[\partial_\mu \left(A_k^T\right) \partial_\mu(\cdot) + \partial_\mu \left(\partial_\mu^T(\cdot)\right) A_k\right], \quad \text{notice } \frac{\partial}{\partial x} = A_1^T \partial_\mu$$

notice that, for the 2rd term

$$\begin{aligned} \sum_{k=1}^{3} A_k^T \partial_\mu \left(\partial_\mu^T(\cdot)\right) A_k &= \operatorname{tr}\left(A^T \partial_\mu \left(\partial_\mu^T(\cdot)\right) A\right) = \operatorname{tr}\left(\partial_\mu \left(\partial_\mu^T(\cdot)\right) AA^T\right) = \operatorname{tr}\left(\partial_\mu \left(\partial_\mu^T(\cdot)\right)\right) \\ &= \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}\right](\cdot) \end{aligned}$$

for the 1st term, write in other form, $A = [a_{ik}] = [A_1, A_2, A_3] = [B_1\ B_2\ B_3]^T$

$$\sum_{k=1}^{3} A_k^T \partial_\mu \left(A_k^T\right) \partial_\mu(\cdot) = \left[\sum_{k=1}^{3} A_k^T \partial_\mu \left(A_k^T\right)\right] \partial_\mu(\cdot)$$

$$\begin{aligned}
\left[\sum_{k=1}^{3} A_k^T \partial_\mu \left(A_k^T\right)\right] &= \left[\sum_{k=1}^{3}\sum_{i=1}^{3} a_{ik}\partial_{\mu i}(a_{1k}) \quad \sum_{k=1}^{3}\sum_{i=1}^{3} a_{ik}\partial_{\mu i}(a_{2k}) \quad \sum_{k=1}^{3}\sum_{i=1}^{3} a_{ik}\partial_{\mu i}(a_{3k})\right] \\
&= \left[\sum_{i=1}^{3}\sum_{k=1}^{3} a_{ik}\partial_{\mu i}(a_{1k}) \quad \sum_{i=1}^{3}\sum_{k=1}^{3} a_{ik}\partial_{\mu i}(a_{2k}) \quad \sum_{i=1}^{3}\sum_{k=1}^{3} a_{ik}\partial_{\mu i}(a_{3k})\right] \\
&= \left[\sum_{i=1}^{3} \partial_{\mu i}(B_1^T)B_i \quad \sum_{i=1} \partial_{\mu i}(B_2^T)B_i \quad \sum_{i=1} \partial_{\mu i}(B_3^T)B_i\right] \\
&= \left[\partial_\mu \left(B_1^T\right)[B_1\ B_2\ B_3] \quad \partial_\mu \left(B_2^T\right)[B_1\ B_2\ B_3] \quad \partial_\mu \left(B_3^T\right)[B_1\ B_2\ B_3]\right] = \left[\frac{2}{r} \quad \frac{\cos\theta}{r\sin\theta} \quad 0\right]
\end{aligned}$$

notice $\frac{\partial}{\partial r}(B_1) = \frac{\partial}{\partial r}(B_2) = \frac{\partial}{\partial r}(B_3) = \frac{\partial}{\partial \theta}(B_3) = \vec{0}$ and $B_i^T B_i = 1 \Rightarrow \frac{\partial}{\partial \theta}(B_2)^T B_2 = \frac{\partial}{\partial \varphi}(B_3)^T B_3 = 0$

$$\partial_\mu \left(B_1^T\right)[B_1\ B_2\ B_3] = \frac{\partial}{\partial r}(B_1)^T B_1 + \frac{1}{r}\frac{\partial}{\partial \theta}(B_1)^T B_2 + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(B_1)^T B_3 = 0 + \frac{1}{r} + \frac{1}{r} = \frac{2}{r}$$

$$\partial_\mu \left(B_2^T\right)[B_1\ B_2\ B_3] = \frac{\partial}{\partial r}(B_2)^T B_1 + \frac{1}{r}\frac{\partial}{\partial \theta}(B_2)^T B_2 + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(B_2)^T B_3 = 0 + 0 + \frac{\cos\theta}{r\sin\theta} = \frac{\cos\theta}{r\sin\theta}$$

$$\partial_\mu \left(B_3^T\right)[B_1\ B_2\ B_3] = \frac{\partial}{\partial r}(B_3)^T B_1 + \frac{1}{r}\frac{\partial}{\partial \theta}(B_3)^T B_2 + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(B_3)^T B_3 = 0 + 0 + 0 = 0$$

to sum up

$$\begin{aligned}
\Delta(\cdot) &= \left[\frac{2}{r} \quad \frac{\cos\theta}{r\sin\theta} \quad 0\right]\left[\frac{\partial}{\partial r} \quad \frac{1}{r}\frac{\partial}{\partial \theta} \quad \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right]^T(\cdot) + \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}\right](\cdot) \\
&= \left[\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}\right](\cdot)
\end{aligned}$$

**step 2** for function $u = u(r)$

$$\Delta u = \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)u = 0$$

Substitute $v \equiv \frac{\mathrm{d}}{\mathrm{d}r}u$, it becomes

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{2}{r}\right)v = 0 \Leftrightarrow \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{2}{r}v$$

(a) $v = 0$, then $u(r) = \int v dr = C$
(b) $v \neq 0$, here $C_1 \neq 0$

$$\ln|v| = \int \frac{\mathrm{d}v}{v} = \int -\frac{2}{r}\mathrm{d}r = -2\ln|r| + C_1' \Leftrightarrow \frac{\mathrm{d}u}{\mathrm{d}r}r^2 = vr^2 = C_1 \Leftrightarrow \mathrm{d}u = C_1\frac{\mathrm{d}r}{r^2}$$

$$u = \int du = \int C_1\frac{\mathrm{d}r}{r^2} = -\frac{C_1}{r} + C_2$$

To sum up (a)(b), all the radically symmetric solution in $\mathbb{R}^3$

$$u(r) = -\frac{C_1}{r} + C_2, \quad C_1, C_2 \in \mathbb{R}$$

# Problem 12

12. Verify the formula 3.6 for the gradient in polar coordinates. Derive the formula for the divergence in polar coordinates

$$\text{grad} = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta \tag{3.6}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit polar vectors
(Section 6.3, Problems 12)

**solution**
notice that

$$x = r\cos\theta$$
$$y = r\sin\theta$$

the relationship between $(\mathbf{e}_r\ \mathbf{e}_\theta)^T$ and $(\mathbf{e}_x\ \mathbf{e}_y)^T$

$$\mathbf{e}_r = \frac{\left[\frac{\partial(x,y)}{\partial(r)}\right]^T}{|\frac{\partial(x,y)}{\partial(r)}|}\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{pmatrix} = \frac{[\cos\theta \quad \sin\theta]}{1}\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{pmatrix} = [\cos\theta \quad \sin\theta]\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{pmatrix}$$

$$\mathbf{e}_\theta = \frac{\left[\frac{\partial(x,y)}{\partial(\theta)}\right]^T}{|\frac{\partial(x,y)}{\partial(\theta)}|}\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{pmatrix} = \frac{[-r\sin\theta \quad r\cos\theta]}{r}\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{pmatrix} = [-\sin\theta \quad \cos\theta]\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{pmatrix}$$

$$\begin{pmatrix}\mathbf{e}_r\\ \mathbf{e}_\theta\end{pmatrix} = \begin{pmatrix}\left[\frac{\partial(x,y)}{\partial(r)}\right]^T/|\frac{\partial(x,y)}{\partial(r)}|\\ \left[\frac{\partial(x,y)}{\partial(\theta)}\right]^T/|\frac{\partial(x,y)}{\partial(\theta)}|\end{pmatrix}\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{pmatrix}$$

similarly

$$\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \theta}\end{pmatrix} = \left[\frac{\partial(x,y)}{\partial(r,\theta)}\right]^T\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix} = \begin{pmatrix}\left[\frac{\partial(x,y)}{\partial(r)}\right]^T\\ \left[\frac{\partial(x,y)}{\partial(\theta)}\right]^T\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}$$

thus

$$\begin{pmatrix}\frac{1}{|\frac{\partial(x,y)}{\partial(r)}|}\frac{\partial}{\partial r}\\ \frac{1}{|\frac{\partial(x,y)}{\partial(\theta)}|}\frac{\partial}{\partial \theta}\end{pmatrix} = \begin{pmatrix}\left[\frac{\partial(x,y)}{\partial(r)}\right]^T/|\frac{\partial(x,y)}{\partial(r)}|\\ \left[\frac{\partial(x,y)}{\partial(\theta)}\right]^T/|\frac{\partial(x,y)}{\partial(\theta)}|\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}$$

define $A$ as below, we can verify $A^{-1} = A^T$

$$A \equiv \begin{pmatrix}\left[\frac{\partial(x,y)}{\partial(r)}\right]^T/|\frac{\partial(x,y)}{\partial(r)}|\\ \left[\frac{\partial(x,y)}{\partial(\theta)}\right]^T/|\frac{\partial(x,y)}{\partial(\theta)}|\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

$$\begin{pmatrix}\mathbf{e}_r\\ \mathbf{e}_\theta\end{pmatrix} = A\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{pmatrix}, \quad \begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial \theta}\end{pmatrix} = \begin{pmatrix}\frac{1}{|\frac{\partial(x,y)}{\partial(r)}|}\frac{\partial}{\partial r}\\ \frac{1}{|\frac{\partial(x,y)}{\partial(\theta)}|}\frac{\partial}{\partial \theta}\end{pmatrix} = A\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}$$

Finally, gradient can be expressed as

$$\nabla = \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}^T\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{pmatrix} = \left[A^{-1}\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial \theta}\end{pmatrix}\right]^T A^{-1}\begin{pmatrix}\mathbf{e}_r\\ \mathbf{e}_\theta\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial \theta}\end{pmatrix}^T AA^{-1}\begin{pmatrix}\mathbf{e}_r\\ \mathbf{e}_\theta\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial \theta}\end{pmatrix}^T\begin{pmatrix}\mathbf{e}_r\\ \mathbf{e}_\theta\end{pmatrix}$$
$$= \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta$$

Now let think about the **divergence**, the vector field $\vec{F}$ can be written in 2 different orthogonal coordinate systems

$$\vec{F} = F^T \mathbf{e} = [F^\mu]^T \mathbf{e}_\mu$$

Where

$$F = \begin{pmatrix} F^x \\ F^y \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{pmatrix} \quad F^\mu = \begin{pmatrix} F^r \\ F^\theta \end{pmatrix} \quad \mathbf{e}^\mu = \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix}$$

Since $\mathbf{e}_\mu = A\mathbf{e}$, and $AA^T = I$

$$F^\mu = AF$$

Then define the notation $\partial, \partial^\mu, A_k, B_i$ $(k, i = 1, 2)$ as

$$\partial = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \quad \partial_\mu = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \end{pmatrix} \quad A = [a_{ik}] = [A_1 \ A_2] = [B_1 \ B_2]^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Consider the definition of the divergence of $\vec{F}$ in Cartesian coordinate system

$$\nabla \cdot \vec{F} \equiv \partial^T F$$

Then consider to replace $\partial, F$ with $\partial_\mu, F^\mu$

$$\partial = A^T \partial_\mu = \begin{pmatrix} A_1^T \partial_\mu \\ A_2^T \partial_\mu \end{pmatrix} \quad F = A^T F^\mu = \begin{pmatrix} A_1^T F^\mu \\ A_2^T F^\mu \end{pmatrix}$$

Thus, we can write in such form (just like what we did to derive the Laplacian in previous problem)

$$\nabla \cdot \vec{F} \equiv \partial^T F = \sum_{k=1}^{2} A_k^T \partial_\mu \left(A_k^T F^\mu\right) = \left[\sum_{k=1}^{2} A_k^T \partial_\mu \left(A_k^T\right)\right] F^\mu + \sum_{k=1}^{2} A_k^T \partial_\mu \left([F^\mu]^T\right) A_k$$

For the 2nd term

$$\sum_{k=1}^{2} A_k^T \partial_\mu \left([F^\mu]^T\right) A_k = \mathrm{tr}\left(A^T \partial_\mu \left([F^\mu]^T\right) A\right) = \mathrm{tr}\left(\partial_\mu \left([F^\mu]^T\right) AA^T\right) = \mathrm{tr}\left(\partial_\mu \left([F^\mu]^T\right)\right) = \frac{\partial}{\partial r} F^r + \frac{1}{r}\frac{\partial}{\partial \theta} F^\theta$$

For the 1st term

$$\left[\sum_{k=1}^{2} A_k^T \partial_\mu \left(A_k^T\right)\right] = \left[\sum_{k=1}^{2}\sum_{i=1}^{2} a_{ik}\partial_{\mu i}(a_{1k}) \quad \sum_{k=1}^{2}\sum_{i=1}^{2} a_{ik}\partial_{\mu i}(a_{2k})\right] = \left[\sum_{i=1}^{2}\sum_{k=1}^{2} a_{ik}\partial_{\mu i}(a_{1k}) \quad \sum_{i=1}^{2}\sum_{k=1}^{2} a_{ik}\partial_{\mu i}(a_{2k})\right]$$

$$= \left[\sum_{i=1}^{3} \partial_{\mu i}(B_1^T) B_i \quad \sum_{i=1}^{2} \partial_{\mu i}(B_2^T) B_i\right] = \left[\partial_\mu \left(B_1^T\right) [B_1 \ B_2] \quad \partial_\mu \left(B_2^T\right) [B_1 \ B_2]\right] = \begin{bmatrix} \frac{1}{r} & 0 \end{bmatrix}$$

notice $\frac{\partial}{\partial r}(B_1) = \frac{\partial}{\partial r}(B_2) = \vec{0}$ and $B_i^T B_i = 1 \Rightarrow \frac{\partial}{\partial \theta}(B_2)^T B_2 = 0$

$$\partial_\mu \left(B_1^T\right) [B_1 \ B_2] = \frac{\partial}{\partial r}(B_1)^T B_1 + \frac{1}{r}\frac{\partial}{\partial \theta}(B_1)^T B_2 = 0 + \frac{1}{r} = \frac{1}{r}$$

$$\partial_\mu \left(B_2^T\right) [B_1 \ B_2] = \frac{\partial}{\partial r}(B_2)^T B_1 + \frac{1}{r}\frac{\partial}{\partial \theta}(B_2)^T B_2 = 0 + 0 = 0$$

In the end

$$\begin{aligned} \nabla \cdot \vec{F} &= \left[\sum_{k=1}^{2} A_k^T \partial_\mu \left(A_k^T\right)\right] F^\mu + \sum_{k=1}^{2} A_k^T \partial_\mu \left([F^\mu]^T\right) A_k \\ &= \begin{bmatrix} \frac{1}{r} & 0 \end{bmatrix} F^\mu + \mathrm{tr}\left(\partial_\mu \left([F^\mu]^T\right)\right) \\ &= \frac{1}{r} F^r + \frac{\partial}{\partial r} F^r + \frac{1}{r}\frac{\partial}{\partial \theta} F^\theta \end{aligned}$$

# Problem 6

6. Organisms of density $u(x,t)$ are distributed in a patch of length $l$. They diffuse with diffusion constant $D$, and their growth rate is $ru$. At the ends of the patch a zero density is maintained, and the initial distribution is $u(x,0)=f(x)$. Because there is competition between growth in the interior of the patch and escape from the boundaries, it is interesting to know whether the population increases or collapses. Show that the condition on the patch size, $l<\pi\sqrt{D/r}$, ensures death of the population
(Section 6.4, Problems 6)

**solution**
The equations of $u(x,t)$ are

$$\frac{\partial u}{\partial t}=ru+D\frac{\partial^2 u}{\partial x^2}$$
$$u(0,t)=u(l,t)=0$$
$$u(x,0)=f(x)$$

Write the solution as the weighted sum of fundamental solutions $X_n(x)T_n(t)$

$$u(x,t)=\sum c_n X_n(x)T_n(t)$$

It leads to

$$\frac{\frac{dT_n}{dt}}{DT_n}-\frac{r}{D}=\frac{\frac{d^2X_n}{dx^2}}{X_n}=-\lambda_n$$

For $T_n(t)$

$$\frac{dT_n}{dt}=-(D\lambda_n-r)T_n\Rightarrow T_n=e^{-(D\lambda_n-r)t}$$

For $X_n(x)$

$$\frac{d^2X_n}{dx^2}=-\lambda_n X_n$$
$$X_n(0)=X_n(l)=0$$
$$X_n=\sin(\sqrt{\lambda_n}x),\ \sqrt{\lambda_n}l=n\pi, n\in\mathbb{Z}^+\Rightarrow\lambda_n=\frac{n^2\pi^2}{l^2},\ X_n=\sin\left(\frac{n\pi}{l}x\right),\ n\in\mathbb{Z}^+$$

$u(x,t)$ can be written as

$$u(x,t)=\sum_{n=1}^{\infty}c_n\sin\left(\frac{n\pi}{l}x\right)e^{-\left(D\frac{n^2\pi^2}{l^2}-r\right)t}$$

With the initial condition

$$u(x,0)=\sum_{n=1}^{\infty}c_n\sin\left(\frac{n\pi}{l}x\right)=f(x)$$

Thus, for the $c_n$

$$c_n=\frac{\int_0^l f(\xi)\sin\left(\frac{n\pi}{l}\xi\right)d\xi}{\int_0^l\sin^2\left(\frac{n\pi}{l}\xi\right)d\xi}=\frac{2}{l}\int_0^l f(\xi)\sin\left(\frac{n\pi}{l}\xi\right)d\xi$$

$$\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l f(\xi)\sin\left(\frac{n\pi}{l}\xi\right)d\xi\right]\sin\left(\frac{n\pi}{l}x\right)e^{-\left(D\frac{n^2\pi^2}{l^2}-r\right)t}\\&=\int_0^l G(x,\xi,t)f(\xi)d\xi\end{aligned}$$

Where $G(x,\xi,t) = \frac{2}{l}\sum_{n=1}^{\infty} \exp(-(D\frac{n^2\pi^2}{l^2} - r)t)\sin(\frac{n\pi}{l}\xi)\sin(\frac{n\pi}{l}x)$
Notice that

$$
\begin{aligned}
|u(x,t)| = \left|\int_0^l G(x,\xi,t)f(\xi)d\xi\right| &\leq \int_0^l |G(x,\xi,t)|\cdot|f(\xi)|\,d\xi \\
&\leq \left[\frac{2}{l}\sum_{n=1}^{\infty} e^{-\left(D\frac{n^2\pi^2}{l^2}-r\right)t}\right]\int_0^l |f(\xi)|d\xi
\end{aligned}
$$

When $l < \pi\sqrt{\frac{D}{r}} \Rightarrow 1 - \frac{rl^2}{D\pi^2} > 0$, and notice $n^2 - 1 \geq 3(n-1),\ n \in \mathbb{Z}^+$

$$
D\frac{n^2\pi^2}{l^2} - r = D\frac{\pi^2}{l^2}(n^2-1) + D\frac{\pi^2}{l^2}(1 - \frac{rl^2}{D\pi^2}) \geq 3D\frac{\pi^2}{l^2}(n-1) + D\frac{\pi^2}{l^2}(1 - \frac{rl^2}{D\pi^2})
$$

$$
\begin{aligned}
\left[\frac{2}{l}\sum_{n=1}^{\infty} e^{-\left(D\frac{n^2\pi^2}{l^2}-r\right)t}\right] &\leq \frac{2}{l}\left[\sum_{n=1}^{\infty} e^{-3D\frac{\pi^2}{l^2}(n-1)t}\right] e^{-D\frac{\pi^2}{l^2}(1-\frac{rl^2}{D\pi^2})t} \\
&= \frac{2}{l}\cdot\frac{1}{1-e^{-3D\frac{\pi^2}{l^2}t}}\cdot e^{-D\frac{\pi^2}{l^2}(1-\frac{rl^2}{D\pi^2})t}
\end{aligned}
$$

We know that

$$
\lim_{t\to\infty}\frac{1}{1-e^{-3D\frac{\pi^2}{l^2}t}} = 1,\ \lim_{t\to\infty} e^{-D\frac{\pi^2}{l^2}(1-\frac{rl^2}{D\pi^2})t} = 0
$$

$$
\lim_{t\to\infty}\left[\frac{2}{l}\sum_{n=1}^{\infty} e^{-\left(D\frac{n^2\pi^2}{l^2}-r\right)t}\right] = \frac{2}{l}\lim_{t\to\infty}\frac{1}{1-e^{-3D\frac{\pi^2}{l^2}t}}\lim_{t\to\infty} e^{-D\frac{\pi^2}{l^2}(1-\frac{rl^2}{D\pi^2})t} = 0
$$

In the end

$$
\lim_{t\to\infty}|u(x,t)| = \lim_{t\to\infty}\left[\frac{2}{l}\sum_{n=1}^{\infty} e^{-\left(D\frac{n^2\pi^2}{l^2}-r\right)t}\right]\cdot\int_0^l |f(\xi)|d\xi = 0\cdot\int_0^l |f(\xi)|d\xi = 0
$$

It shows that the condition on the patch size, $l < \pi\sqrt{\frac{D}{r}}$, ensures death of the population

## Problem 7

7. Consider Laplace's equation on the unit circle with given boundary condition:

$$r\left(ru_r\right)_r + u_{\theta\theta} = 0, \quad 0 < r < 1, 0 < \theta \le 2\pi$$
$$u(1,\theta) = f(\theta), \quad 0 < \theta \le 2\pi$$

(Section 6.4, Problems 7)

a) Assuming $u = R(r)Y(\theta)$, along with the implicit periodic boundary conditions $u(r,0) = u(r,2\pi), u_\theta(r,0) = u_\theta(r,2\pi)$, use separation of variables to find an infinite series representation of the solution
b) Show that the solution can be written in integral for as

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(1-r^2\right) f(\phi)}{1+r^2-2r\cos(\theta-\phi)} d\phi$$

This representation is **Poisson's integral formula**

**solution**

Write the solution as sum of fundamental solutions

$$u(r,\theta) = \sum R_n(r)Y_n(\theta)$$

It leads to

$$\frac{r\frac{d}{dr}(r\frac{dR_n}{dr})}{R_n} = -\frac{\frac{d^2Y_n}{d\theta^2}}{Y_n} = \lambda_n$$

For $Y_n(\theta)$

$$\frac{d^2Y_n}{d\theta^2} + \lambda_n Y_n = 0$$

$$Y_n(0) = Y_(2\pi), \quad \frac{dY_n}{d\theta}|_{\theta=0} = \frac{dY_n}{d\theta}|_{\theta=2\pi}$$

The characteristic equation $r^2 + \lambda_n = 0$
(i) $r_1 = r_2 \Rightarrow \lambda_0 = 0$, then $Y_0(0) = Y_0(2\pi) \Rightarrow Y_0(\theta) = A_0,\ A_0 \in \mathbb{R}$
(ii) $r_1 \ne r_2 \Rightarrow \lambda_n \ne 0$

$$Y_n(0) = Y_(2\pi), \quad \frac{dY_n}{d\theta}|_{\theta=0} = \frac{dY_n}{d\theta}|_{\theta=2\pi} \Rightarrow Y_n = c_n\cos(\sqrt{\lambda_n}\theta) + d_n\sin(\sqrt{\lambda_n}\theta)$$

$$\sqrt{\lambda_n}\cdot 2\pi = n\cdot 2\pi \Rightarrow \lambda_n = n^2, Y_n = A_n e^{in\theta} + A_n^* e^{-in\theta},\ n \in \mathbb{Z}^+$$

Where $A_n = \frac{c_n}{2} - i\frac{d_n}{2}, A_n^* = \frac{c_n}{2} + i\frac{d_n}{2}$ are a conjugate pair, $A_n \in \mathbb{C}$
For $R_n(r)$

$$r\frac{d}{dr}(r\frac{dR_n}{dr}) - \lambda_n R_n = r^2\frac{d^2R_n}{dr^2} + r\frac{dR_n}{dr} - n^2R_n = 0$$

The indicial equation of Euler-Cauchy equation is $m^2 - n^2 = 0$
(i) $m_1 = m_2 \Rightarrow n = 0$, then $m_{1,2} = 0,\ R_0 = c_1' + c_2'\ln(r)$, assume $u$ are bounded $\Rightarrow R_0 = 1$
(ii) $m_1 \ne m_2 \Rightarrow n \ne 0$, then $m_{1,2} = \pm n,\ R_0 = c_1' r^n + c_2' r^{-n}$, assume $u$ are bounded $\Rightarrow R_n = r^n$
Thus, the expression of $u(r,\theta)$, $A_0 \in \mathbb{R}, A_n \in \mathbb{C}, n \in \mathbb{Z}^+$

$$u(r,\theta) = \sum R_n(r)Y_n(\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n e^{in\theta} + A_n^* e^{-in\theta}\right] r^n$$

When $r = 1$, with the boundary condition

$$u(1,\theta) = A_0 + \sum_{n=1}^{\infty} \left[A_n e^{in\theta} + A_n^* e^{-in\theta}\right] = f(\theta)$$

For $A_0$, notice $\int_0^{2\pi} e^{in\phi} d\phi = \int_0^{2\pi} e^{-in\phi} d\phi = 0$

$$\int_0^{2\pi} f(\phi) d\phi = \int_0^{2\pi} A_0 d\phi = 2\pi \cdot A_0$$

For $A_n$, notice $\int_0^{2\pi} e^{i(n'-n)\phi} d\phi = 0$ when $n' \neq n$, and $\int_0^{2\pi} e^{-i(n'+n)\phi} d\phi = 0$

$$\int_0^{2\pi} f(\phi) e^{-in\phi} d\phi = \int_0^{2\pi} A_n d\phi = 2\pi \cdot A_n$$

To sum up

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi,\ A_n = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{-in\phi} d\phi,\ A_n^* = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{in\phi} d\phi,$$

For $u(r,\theta)$, where $r < 1$

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \left[1 + \sum_{n=1}^{\infty} (e^{-in\phi} e^{in\theta} + e^{in\phi} e^{-in\theta}) r^n\right] d\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \left[1 + \sum_{n=1}^{\infty} [re^{i(\theta-\phi)}]^n + [re^{-i(\theta-\phi)}]^n\right] d\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \left[1 + \frac{re^{i(\theta-\phi)}}{1 - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{1 - re^{-i(\theta-\phi)}}\right] d\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \left[1 + \frac{2r\cos(\theta-\phi) - 2r^2}{1 + r^2 - 2r\cos(\theta-\phi)}\right] d\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)\, f(\phi)}{1 + r^2 - 2r\cos(\theta-\phi)} d\phi
\end{aligned}$$

In the end, **Poisson's integral formula** is proved