## Problem 3

3. Verify the equilibria and their stability for the basic model 6.1, 6.2, and 6.3 as discussed in the text

$$\frac{dV}{d\tau} = aY - d_v V \tag{6.1}$$

$$\frac{dX}{d\tau} = c - d_x X - \beta XV \tag{6.2}$$

$$\frac{dY}{d\tau} = \beta XV - d_y Y \tag{6.3}$$

(Problems 3 on Page 132, PDF Page 159)

**solution**

Select the characteristic scaling: $\tau_c = \frac{1}{d_x}, V_c = \frac{ac}{d_v d_y}, X_c = Y_c = \frac{c}{d_x}$
define $t \equiv \tau/\tau_c, v \equiv V/V_c, x \equiv X/X_c, y \equiv Y/Y_c$, the model 6.1, 6.2, 6.3 becomes

$$\frac{dv}{dt} = \frac{\left(\frac{d_y}{d_x}\right)}{\left(\frac{d_x}{d_v}\right)} y - \frac{1}{\left(\frac{d_x}{d_v}\right)} v$$

$$\frac{dx}{dt} = 1 - x - \left[\frac{\beta}{d_x}\frac{ac}{d_v d_y}\right] xv$$

$$\frac{dy}{dt} = \left[\frac{\beta}{d_x}\frac{ac}{d_v d_y}\right] xv - \left(\frac{d_y}{d_x}\right) y$$

Now define $\varepsilon \equiv \left(\frac{d_x}{d_v}\right), \alpha \equiv \left(\frac{d_y}{d_x}\right), R_0 \equiv \left[\frac{\beta}{d_x}\frac{ac}{d_v d_y}\right]$, equations are equivalent to

$$\varepsilon \frac{dv}{dt} = \alpha y - v$$

$$\frac{dx}{dt} = 1 - x - R_0 xv$$

$$\frac{dy}{dt} = R_0 xv - \alpha y$$

Let $\frac{dv}{dt} = \frac{dx}{dt} = \frac{dy}{dt} = 0$ to find the critical points

$$v = \alpha y$$

$$R_0 vx = 1 - x$$

$$R_0 vx = \alpha y$$

Thus, so with $v = \alpha y, x = 1 - \alpha y$, moreover because of $R_0 > 0, \alpha > 0$

$$R_0 \alpha y(1 - \alpha y) = \alpha y \Leftrightarrow y\left[R_0(1 - \alpha y) - 1\right] = 0 \Leftrightarrow y = 0 \text{ or } \frac{1 - \frac{1}{R_0}}{\alpha}$$

For $y = 0$, the critical point $(v^*, x^*, y^*) = (0, 1, 0)$

For $y = \frac{1-\frac{1}{R_0}}{\alpha}$, the critical point $(v^*, x^*, y^*) = (1 - \frac{1}{R_0}, \frac{1}{R_0}, \frac{1-\frac{1}{R_0}}{\alpha})$ only when $R_0 > 1$

Consider the Jacobian of $\frac{\mathrm{d}}{\mathrm{d}t}(v,x,y) = (P,Q,R)$

$$J \equiv \frac{\partial(P,Q,R)}{\partial(v,x,y)} = \begin{pmatrix} -\frac{1}{\varepsilon} & 0 & \frac{\alpha}{\varepsilon} \\ -R_0 x & -1-R_0 v & 0 \\ R_0 x & R_0 v & -\alpha \end{pmatrix}$$

At the critical point $(v^*,x^*,y^*) = (0,1,0)$

$$J = \begin{pmatrix} -\frac{1}{\varepsilon} & 0 & \frac{\alpha}{\varepsilon} \\ -R_0 & -1 & 0 \\ R_0 & 0 & -\alpha \end{pmatrix}, \quad |\lambda I - J| = \begin{vmatrix} \lambda+\frac{1}{\varepsilon} & 0 & -\frac{\alpha}{\varepsilon} \\ R_0 & \lambda+1 & 0 \\ -R_0 & 0 & \lambda+\alpha \end{vmatrix} = \left[\lambda^2 + (\frac{1}{\varepsilon}+\alpha)\lambda + \frac{\alpha}{\varepsilon}(1-R_0)\right](\lambda+1)$$

eigenvalues $\lambda_3 = -1$, $\lambda_1+\lambda_2 = -(\frac{1}{\varepsilon}+\alpha) < 0, \lambda_1\lambda_2 = \frac{\alpha}{\varepsilon}(1-R_0)$, critical point is stable $\Leftrightarrow \lambda_1\lambda_2 \geq 0$

(1) $0 < R_0 \leq 1, \lambda_1\lambda_2 \geq 0$, the critical point $(v^*,x^*,y^*) = (0,1,0)$ is **stable**
(2) $1 < R_0, \lambda_1\lambda_2 < 0$, the critical point $(v^*,x^*,y^*) = (0,1,0)$ is **unstable**

At the critical point $(v^*,x^*,y^*) = (1-\frac{1}{R_0}, \frac{1}{R_0}, \frac{1-\frac{1}{R_0}}{\alpha})$ only when $R_0 > 1$

$$J = \begin{pmatrix} -\frac{1}{\varepsilon} & 0 & \frac{\alpha}{\varepsilon} \\ -1 & -R_0 & 0 \\ 1 & R_0-1 & -\alpha \end{pmatrix}, \quad |\lambda I - J| = \begin{vmatrix} \lambda+\frac{1}{\varepsilon} & 0 & -\frac{\alpha}{\varepsilon} \\ 1 & \lambda+R_0 & 0 \\ -1 & -R_0+1 & \lambda+\alpha \end{vmatrix} = \lambda\left(\lambda+(\frac{1}{\varepsilon}+\alpha)\right)(\lambda+R_0) + \frac{\alpha}{\varepsilon}(R_0-1)$$

We can prove $\lambda_3 < 0, \mathrm{Re}(\lambda_2) \leq \mathrm{Re}(\lambda_1) < 0$, the critical point is **stable** when it exists ($R_0 > 1$)
method: **Routh-Hurwitz stability criterion**
write the **Routh table** for $\lambda^3 + (R_0 + \frac{1}{\varepsilon} + \alpha)\lambda^2 + (\frac{1}{\varepsilon}+\alpha)R_0\lambda + \frac{\alpha}{\varepsilon}(R_0-1)$

| | | |
|---|---|---|
| $\lambda^3$ | $1$ | $(\frac{1}{\varepsilon}+\alpha)R_0$ |
| $\lambda^2$ | $(R_0+\frac{1}{\varepsilon}+\alpha)$ | $\frac{\alpha}{\varepsilon}(R_0-1)$ |
| $\lambda^1$ | $\frac{[(1+\frac{\alpha\varepsilon}{2})^2+\frac{3}{4}(\alpha\varepsilon)^2]\varepsilon^{-2}R_0+(\frac{1}{\varepsilon}+\alpha)R_0^2+\frac{\alpha}{\varepsilon}}{R_0+\frac{1}{\varepsilon}+\alpha}$ | $0$ |

Number of eigenvalues that have positive real part=Number of sign changes in the first column=**0**
As $R_0 \equiv \left[\frac{\beta}{d_x}\frac{ac}{d_v d_y}\right]$ increases from low to high, the equilibrium of $(V^*,X^*,Y^*)$ changes as follows:

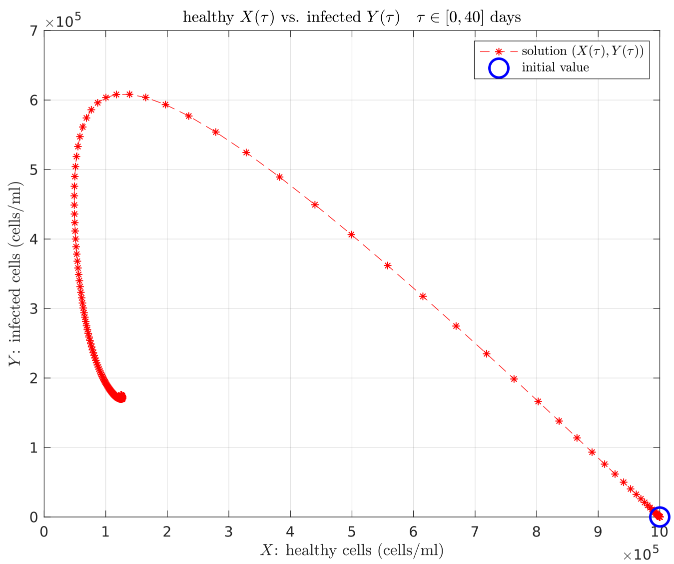
- For $0 < R_0 \equiv \left[\frac{\beta}{d_x}\frac{ac}{d_v d_y}\right] \leq 1$
  **stable** critical point $(v^*,x^*,y^*) = (0,1,0) \Leftrightarrow$ equilibria $(V^*,X^*,Y^*) = (0,X_c,0) = (0,\frac{c}{d_x},0)$
- For $1 < R_0 \equiv \left[\frac{\beta}{d_x}\frac{ac}{d_v d_y}\right]$
  **stable** critical point $(v^*,x^*,y^*) = (1-\frac{1}{R_0}, \frac{1}{R_0}, \frac{1-\frac{1}{R_0}}{\alpha})$
  $\Leftrightarrow$ equilibria $(V^*,X^*,Y^*) = \left((1-\frac{1}{R_0})V_c, \frac{1}{R_0}X_c, \frac{1-\frac{1}{R_0}}{\alpha}Y_c\right) = (\frac{ac}{d_v d_y} - \frac{d_x}{\beta}, \frac{d_v d_y}{\beta a}, \frac{c}{d_y} - \frac{d_x d_v}{\beta a})$

## Problem 4

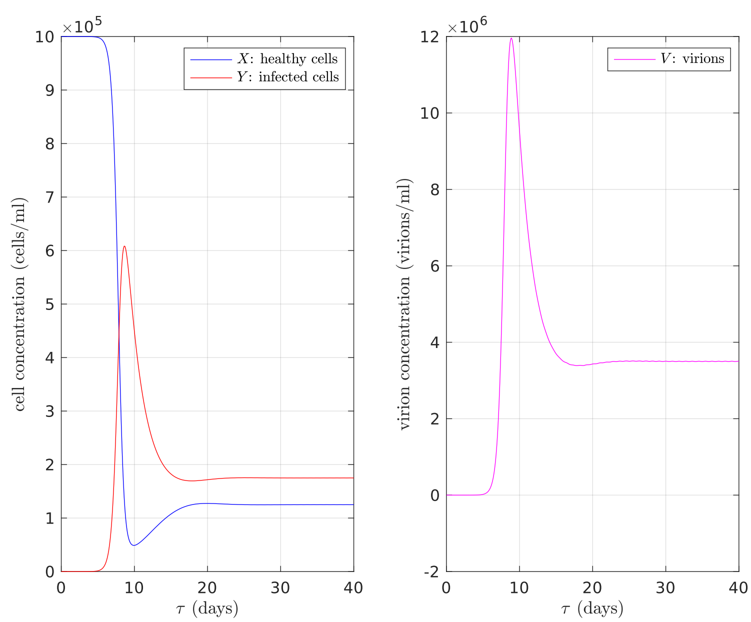
4. Use a numerical differential equation solver (e.g., MATLAB) to obtain a numerical solution to 6.1- 6.3 with the values of the parameters given in the table. For initial conditions take $X(0) = 10^6, Y(0) = 0$, and $V(0) = 1$ (Problems 4 on Page 132, PDF Page 159)

| parameter | $a$ | $d_v$ | $c$ | $d_x$ | $d_y$ | $\beta$ |
|---|---|---|---|---|---|---|
| dimensions | $\mathrm{T}^{-1}$ | $\mathrm{T}^{-1}$ | $\text{cells} \cdot \text{ml}^{-1}\mathrm{T}^{-1}$ | $\mathrm{T}^{-1}$ | $\mathrm{T}^{-1}$ | $(\text{cells/ml})^{-1}\,\mathrm{T}^{-1}$ |
| value range | 100 | 5 | $10^5$ | 0.1 | 0.5 | $2 \cdot 10^{-7}$ |

### solution




(A) Trajectory of $(X(\tau), Y(\tau))$ for 40 days




(B) Left: $X(\tau)$ and $Y(\tau)$; Right: $V(\tau)$

Results: experimental & theoretical results for equilibria

here $R_0 \equiv \left[\frac{\beta}{d_x}\frac{ac}{d_v d_y}\right] = 8 > 1$, so the stable equilibria $(V^*, X^*, Y^*) = (\frac{ac}{d_v d_y} - \frac{d_x}{\beta}, \frac{d_v d_y}{\beta a}, \frac{c}{d_y} - \frac{d_x d_v}{\beta a})$

We can see the experimental & theoretical results for equilibria are very close

```
numerical    (V*,X*,Y*)=(3.503e+06, 1.25e+05, 1.75e+05)
theoretical  (V*,X*,Y*)=(3.5e+06, 1.25e+05, 1.75e+05)
```

Codes listed below

```
clear; clc; close all
% parameters
a = 100; c = 1e5; beta = 2e-7;
d_v = 5; d_x = 0.1; d_y = 0.5;
% characteristic scaling
tau_c = 1 / d_x; V_c = a * c / (d_v * d_y);
X_c = c / d_x; Y_c = c / d_x;
% intial value && end time
V0=1; X0=1e6; Y0=0;
v0 = V0 / V_c; x0 = X0 / X_c; y0 = Y0 / Y_c;
tau_end = 40; t_end = tau_end / tau_c;
% parameters in equation
epsilon = d_x / d_v;
alpha = d_y / d_x;
R_0 = (beta / d_x) * (a * c / (d_v * d_y));
% solve differential equation
dvdt = @(v, x, y) (alpha * y - v) / epsilon;
dxdt = @(v, x, y) 1 - x - R_0 * x * v;
dydt = @(v, x, y) R_0 * x * v - alpha * y;
func = @(t, sol) [dvdt(sol(1,:), sol(2,:), sol(3,:)); ...
                  dxdt(sol(1,:), sol(2,:), sol(3,:)); ...
                  dydt(sol(1,:), sol(2,:), sol(3,:))];
tspan = (0:0.01:t_end)';
[t, solution] = ode23(func, tspan, [v0 x0 y0]);
% rescaling back
[v, x, y] = deal(solution(:, 1), solution(:, 2), solution(:, 3));
[tau, V, X, Y] = deal(t * tau_c, v * V_c, x * X_c, y * Y_c);
% plot X vs. Y
plot(X, Y, 'r--*');
xlabel('$X$: healthy cells (cells/ml)','Interpreter','latex');
ylabel('$Y$: infected cells (cells/ml)','Interpreter','latex');
title(['healthy $X(\tau)$ vs. infected $Y(\tau)\quad \tau\in[0,',...
num2str(tau_end),']$ days'],'Interpreter','latex');
hold on; plot(X0, Y0, 'bo','MarkerSize',15,'LineWidth',2); grid on;
leg = legend('solution $(X(\tau), Y(\tau))$','initial value');
set(leg,'Interpreter','latex');
% plot subfigures for tau ~ V, X, Y
figure(); subplot(1,2,1)
plot(tau, X, 'b-', tau, Y, 'r'); grid on
xlabel('$\tau$ (days)','Interpreter','latex'); xlim([0, tau_end]);
```

```
ylabel('cell concentration (cells/ml)','Interpreter','latex');
leg1 = legend('$X$: healthy cells','$Y$: infected cells');
set(leg1,'Interpreter','latex');
subplot(1,2,2); plot(tau, V, 'm'); grid on
xlabel('$\tau$ (days)','Interpreter','latex'); xlim([0, tau_end]);
ylabel('virion concentration (virions/ml)','Interpreter','latex');
leg2 = legend('$V$: virions');
set(leg2,'Interpreter','latex');
% display the equilibria: numerical vs. theoretical
fprintf('numerical   (V*,X*,Y*)=(%.4g, %.4g, %.4g)\n', ...
V(end),X(end),Y(end));
V_eq = (a * c) / (d_v * d_y) - d_x / beta;
X_eq = d_v * d_y / (beta * a);
Y_eq = c / d_y - d_x * d_v / (beta * a);
fprintf('theoretical (V*,X*,Y*)=(%.4g, %.4g, %.4g)\n',V_eq,X_eq,Y_eq);
```

## Problem 6

6. In an SIR epidemic with $R_0 > 1$, where $R_0 z$ is small for all time, use 6.13 to show, approximately,

$$\frac{dz}{d\tau} = 1 - z - x_0 e^{-R_0 z} \tag{6.13}$$

$$\frac{dz}{d\tau} = (R_0 - 1)\, z \left(1 - \frac{z}{2\,(R_0 - 1)\,/R_0^2}\right)$$

Note that this is the logistic equation. Sketch a graph of the rate of removal $dz/d\tau$ vs. $\tau$. For example, in the plague, the removal rate closely approximates the death rate.

(Problem 6 on Page 139, PDF Page 166)

**solution**

Here after scaling $x = \frac{S}{N}, \quad y = \frac{I}{N}, \quad z = \frac{R}{N}, \quad \tau = \frac{t}{r^{-1}}$
the model starts from initial point $(x, y, z) = (x_0, 1 - x_0, 0)$, assume the number of individuals that are susceptible to the illness $S(t)|_{t=0} \approx N$ is almost the total number at the beginning

$$x_0 = \frac{S(t)|_{t=0}}{N} \approx 1 \Rightarrow \frac{dz}{d\tau} \approx 1 - z - e^{-R_0 z}$$

Expand $e^{-R_0 z}$ to get the approximation, where $R_0 z$ is small for all time

$$e^{-R_0 z} = 1 - R_0 z + \frac{1}{2!}(R_0 z)^2 + \cdots \approx 1 - R_0 z + \frac{1}{2} R_0^2 z^2$$

Thus, substitute $e^{-R_0 z} \approx 1 - R_0 z + \frac{1}{2} R_0^2 z^2$ in the equation

$$\frac{dz}{d\tau} \approx 1 - z - \left[1 - R_0 z + \frac{1}{2} R_0^2 z^2\right] = (R_0 - 1)z - \frac{1}{2} R_0^2 z^2 = (R_0 - 1)z \left(1 - \frac{z}{2(R_0 - 1)/R_0^2}\right)$$

Try to solve the separable equation

$$\ln\left(\frac{\frac{z}{2(R_0-1)/R_0^2}}{1 - \frac{z}{2(R_0-1)/R_0^2}}\right) = \int \frac{d\frac{z}{2(R_0-1)/R_0^2}}{\left(\frac{z}{2(R_0-1)/R_0^2}\right)\left(1 - \frac{z}{2(R_0-1)/R_0^2}\right)} = \int (R_0 - 1) d\tau = (R_0 - 1)(\tau - \tau_0)$$

Thus

$$z(\tau) = \frac{2(R_0 - 1)}{R_0^2} \frac{1}{1 + e^{-(R_0 - 1)(\tau - \tau_0)}}$$

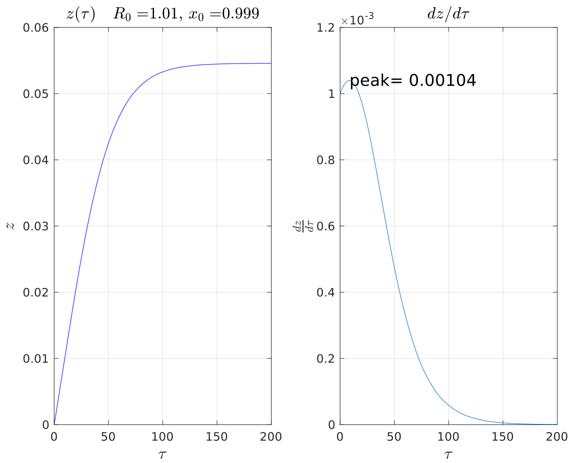
The only useful information from above equation is that $z$ vs. $\tau$ is like logistic curve:
(i) it has bound $M < 1$, for all $z$, holds $z < M$
(ii) $\frac{dz}{d\tau} > 0$, and has peak value at $\tau = \tau_{peak}$ when $1 - x_0$ is small enough
But notice that:
(i) We can not estimate the limit of $z$ with $R_0$ only, it is sensitive to $x_0$

$$\lim_{\tau \to \infty} z(\tau) \neq \frac{2(R_0 - 1)}{R_0^2}$$

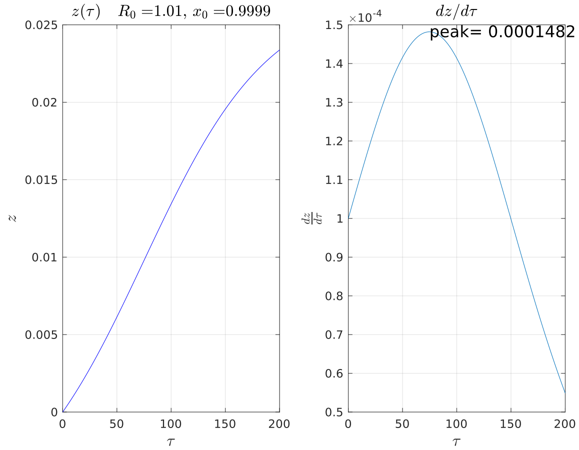
(ii) We can not estimate the max value $\frac{dz}{d\tau}|_{\tau = \tau_{peak}}$ with $R_0$ only, it is sensitive to $x_0$

$$\frac{dz}{d\tau}|_{\tau = \tau_{peak}} \neq \frac{(R_0 - 1)^2}{2R_0^2}$$

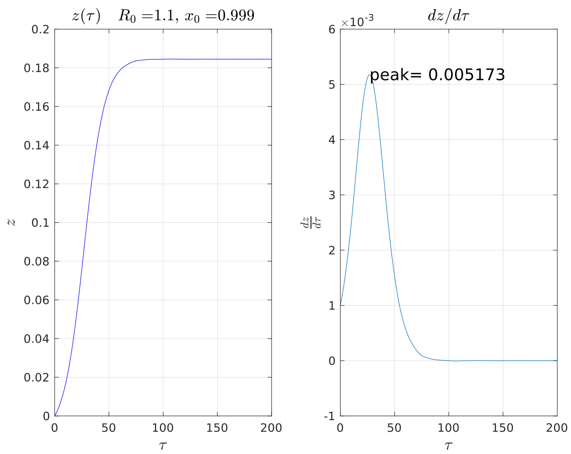
(iii) If $1 - x_0$ is not very small, there is no peak for $\frac{dz}{d\tau}$, for example $R_0 = 1.1,\ x_0 = 0.9$



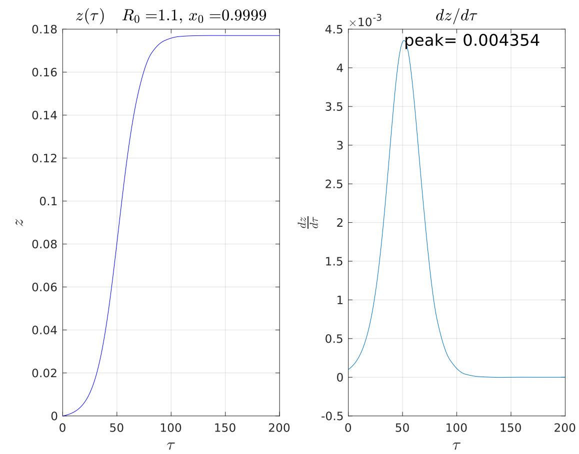

(A) $R_0 = 1.01,\ x_0 = 0.999$



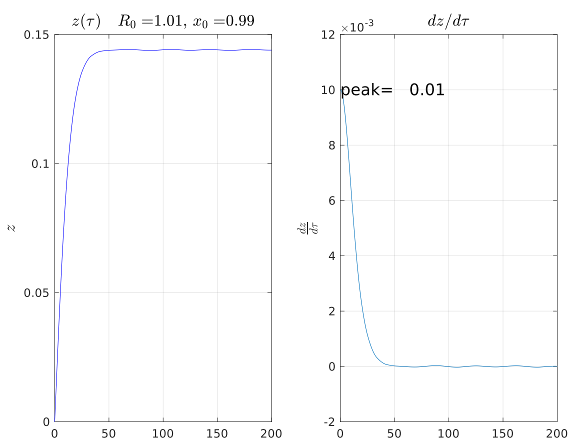

(B) $R_0 = 1.01,\ x_0 = 0.9999$



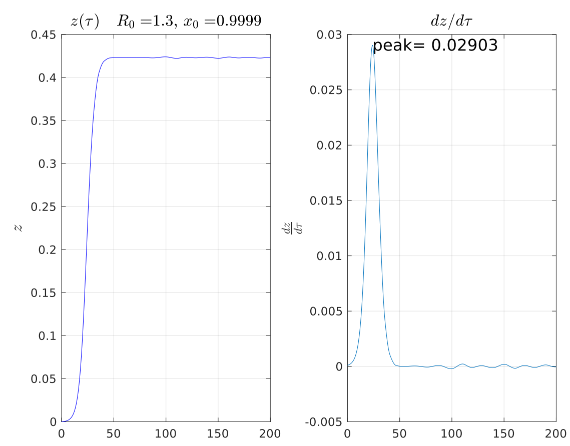

(C) $R_0 = 1.1,\ x_0 = 0.999$




(D) $R_0 = 1.1,\ x_0 = 0.9999$




(E) $R_0 = 1.01,\ x_0 = 0.99$




(F) $R_0 = 1.3,\ x_0 = 0.9999$

FIGURE 2. the diagram of $p$, $\frac{dp}{d\tau}$

```
clear; clc; close all
% solve z(tau) with ode23()
R_0 =  1.3; x_0  = 0.9999; % prameters: R_0, x_0
func = @(tau, z) 1 - z  - x_0 * exp(-R_0 * z);
z0 = 0; dtau = 0.001;
tspan = (0:dtau:200)';
[tau, z] = ode23(func, tspan, z0);
subplot(1, 2, 1); plot(tau,z,'b-'); grid on;
xlabel('$\tau$','Interpreter','latex','FontSize', 16);
ylabel('$z$','Interpreter','latex','FontSize', 16);
title(['$z(\tau) \quad R_0=$',num2str(R_0),',\ $x_0=$',...
num2str(x_0)],'Interpreter','latex','FontSize', 16);
% calc  dz / dtau, display peak of dz / dtau
dzdtau = func(tau, z);
subplot(1, 2, 2); plot(tau, dzdtau); grid on;
xlabel('$\tau$','Interpreter','latex','FontSize', 16);
ylabel('$\frac{dz}{d\tau}$','Interpreter','latex','FontSize', 16);
title('$dz / d\tau$',...
'Interpreter','latex','FontSize', 16);
[dzdtau_peak, ind] = max(dzdtau); % find peak of dz / dtau
tau_peak = tau(ind);
text(tau_peak, dzdtau_peak, ...
sprintf('peak= %6.4g', dzdtau_peak),'FontSize', 16);
```

## Problem 2

2. (Malaria) In this exercise develop and analyze a simplified version of the malaria model under the condition that $r$ is much less than $\mu$ (Problem 2 on Page 143, PDF Page 172)

$$\frac{dh}{dt} = ab\left(\frac{M_T}{H_T}\right)m(1-h) - rh \tag{6.14}$$

$$\frac{dm}{dt} = ach(1-m) - \mu m \tag{6.15}$$

a) Beginning with 6.14-6.15, nondimensionalize these equations by rescaling time by taking $\tau = \mu t$. Obtain

$$\begin{aligned} \tfrac{dh}{d\tau} &= \lambda m(1-h) - \varepsilon h \\ \tfrac{dm}{d\tau} &= \eta h(1-m) - m \end{aligned}$$

where

$$\varepsilon = \frac{r}{\mu}, \quad \lambda = \frac{ab}{\mu}\frac{M_T}{H_T}, \quad \eta = \frac{ac}{\mu}$$

b) Assuming $\varepsilon$ is very small, neglect the $\varepsilon h$ term in the host equation and draw the phase portrait. Include the equilibria, nullclines, direction field, and a local stability analysis for the equilibria

c) For the simplified dimensionless model in part (b), with the values given in Table 1, specifically, $a$=0.5, $r$=0.01, and $\mu$=0.5, use a numerical method to draw time series plots of $h$ and $m$ for various initial conditions

Table 1. Sample malaria parameter values

| Parameter | Name | Sample Value |
|---|---|---|
| $M_T/H_T$ | population ratio | 2 |
| $a$ | biring rate | $0.2 - 0.5$ per day |
| $b$ | effective bites infecting humans | 0.5 |
| $c$ | effective bites infecting mosquitos | 0.5 |
| $r$ | recovery rate | $0.01 - 0.05$ per day |
| $\mu$ | monality rate | $0.05 - 0.5$ per day |

**solution**

a) Take the time scaling $\tau \equiv t/(\frac{1}{\mu})$

$$\begin{aligned} \mu\tfrac{dh}{d\tau} &= ab\left(\tfrac{M_T}{H_T}\right)m(1-h) - rh \\ \mu\tfrac{dm}{d\tau} &= ach(1-m) - \mu m \end{aligned}$$

That is

$$\begin{aligned} \tfrac{dh}{d\tau} &= \tfrac{ab}{\mu}\tfrac{M_T}{H_T}m(1-h) - \tfrac{r}{\mu}h \\ \tfrac{dm}{d\tau} &= \tfrac{ac}{\mu}h(1-m) - m \end{aligned}$$

Then define

$$\varepsilon \equiv \frac{r}{\mu}, \quad \lambda \equiv \frac{ab}{\mu}\frac{M_T}{H_T}, \quad \eta \equiv \frac{ac}{\mu}$$

In the end, we derive

$$\begin{aligned} \tfrac{dh}{d\tau} &= \lambda m(1-h) - \varepsilon h \\ \tfrac{dm}{d\tau} &= \eta h(1-m) - m \end{aligned}$$

b) Assuming $\varepsilon$ is very small, neglect the $\varepsilon h$ term

$$\frac{dh}{d\tau} = \lambda m(1-h)$$
$$\frac{dm}{d\tau} = \eta h(1-m) - m$$

Set $\frac{dh}{d\tau} = 0$, $\frac{dm}{d\tau} = 0$ to get $h$ nullclines and $m$ nullclines respectively
$h$ nullclines

$$0 = \lambda m(1-h) \Leftrightarrow m = 0,\ h = 1$$

$m$ nullclines

$$0 = \eta h(1-m) - m \Leftrightarrow h = \frac{1}{\eta}\left(-1 + \frac{1}{1-m}\right)$$

Combine $h$ nullclines and $m$ nullclines, and notice $m \geq 0, h \geq 0$
the critical points

$$(h^*, m^*) = (0, 0), \quad (1, 1 - \frac{1}{\eta+1})$$

sketch the phase portrait

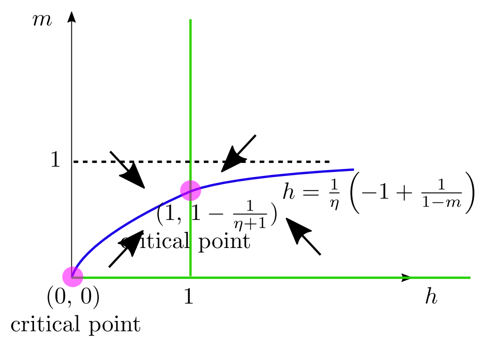
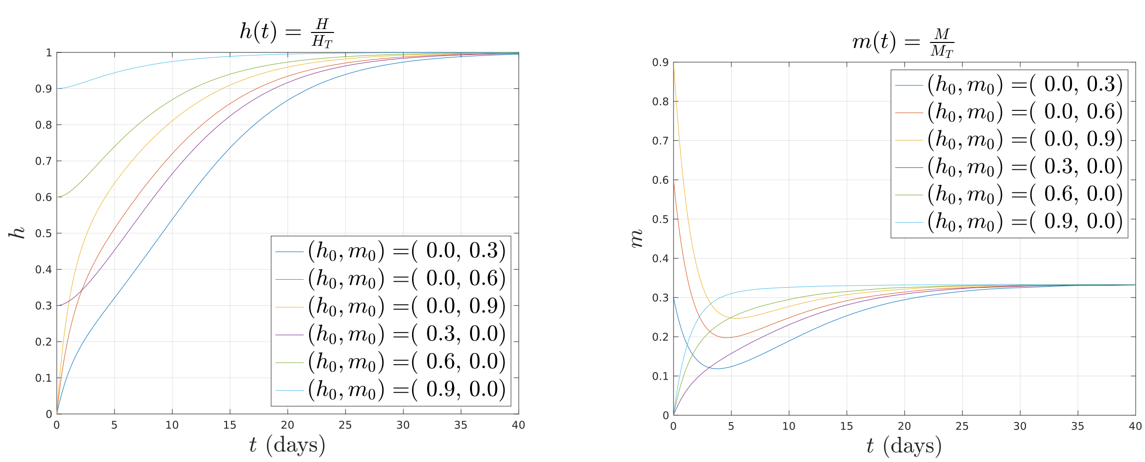



FIGURE 3. The phase portrait

(0, 0) is unstable, the only **stable** equilibria

$$(h^*, m^*) = (1, 1 - \frac{1}{\eta+1}) = (1, \frac{ac}{ac+\mu})$$

c) Here are the time series plots of $h$ and $m$ for various initial conditions $(h_0, m_0)$




(A) $h$ vs. $t$ (B) $m$ vs. $t$

FIGURE 4. The time series plots of $h$ and $m$ vs. $t$

```
clc; clear; close all
% prameters
MT_HT = 2;
a = 0.5; b = 0.5; c= 0.5;
r = 0.01; mu = 0.5;
% coefficients of equations
eps = r / mu;
lambda = (a*b / mu) * MT_HT;
eta = a * c / mu;
t_c = 1 / mu;  % time chateristic scale
% solution initial value (h_0, m_0)
t_end = 40;
tau_end = t_end / t_c;
Sol_init = [0, 0.3;
            0, 0.6;
            0, 0.9;
            0.3, 0;
            0.6, 0;
            0.9, 0]';
% equations
dhdtau = @(h, m) lambda * m.* (1-h);
dmdtau = @(h, m) eta * h.* (1-m) - m;
func = @(tau, sol) [dhdtau(sol(1,:), sol(2,:)); ...
dmdtau(sol(1,:), sol(2,:))];
tspan = (0:0.01:tau_end)';
ax_h = figure(); ax_m = figure();
list_legend = []; handle_h = [];
for sol_init = Sol_init
        [h_0, m_0] = deal(sol_init(1), sol_init(2));
        [tau, solution] = ode23(func, tspan, [h_0, m_0]);
        [t, h, m] = deal(tau * t_c, solution(:, 1), solution(:, 2));
        figure(ax_h); handle = plot(t, h); hold on;
        figure(ax_m); handle = plot(t, m); hold on;
        list_legend=[list_legend;sprintf('$(h_0, m_0)=$...
        (%4.1f, %4.1f)',h_0,m_0)];
end
figure(ax_h);
leg=legend(list_legend,'Location','Southeast');
set(leg,'Interpreter','latex','FontSize', 24);
grid on; xlabel('$t$ (days)','Interpreter','latex','FontSize', 24);
ylabel('$h$','Interpreter','latex','FontSize', 24);
title('$h(t)=\frac{H}{H_T}$','Interpreter','latex','FontSize', 24);
figure(ax_m);
leg=legend(list_legend); set(leg,'Interpreter','latex','FontSize',24);
grid on; xlabel('$t$ (days)','Interpreter','latex','FontSize', 24);
ylabel('$m$','Interpreter','latex','FontSize', 24);
title('$m(t)=\frac{M}{M_T}$','Interpreter','latex','FontSize', 24);
```