Problem 9

9. (6.5.1 on Pages 422–423, PDF page 489)
A toxic chemical diffuses into a semi-infinite domain $x \geq 0$ from its boundary $x = 0$, where the concentration is maintained at g(t). The model is

$$u_t = Du_{xx}, \quad x > 0, t > 0$$
$$u(x, 0) = 0, \quad x > 0$$
$$u(0, t) = g(t), \quad t > 0$$

Determine the concentration $u = u(x, t)$ and write the solution in the form of
$u(x, t) = \int_0^t K(x, t - \tau)g(\tau)d\tau$, identifying the kernel K

solution
Do the Laplace transform for t

$$\mathcal{L}[u(x,t)] = U(x,s),\ \mathcal{L}[\frac{\partial u(x,t)}{\partial t}] = sU(x,s) - u(x,t)|_{t=0} = sU(x,s)$$
$$\mathcal{L}[\frac{\partial^2 u(x,t)}{\partial x^2}] = \frac{\partial^2 U(x,s)}{\partial x^2},\ \mathcal{L}[u(x,t)|_{x=0}] = U(x,s)|_{x=0} = G(s)$$

It leads to

$$\frac{\partial^2 U(x,s)}{\partial x^2} = \frac{s}{D}U(x,s)$$
$$U(x,s)|_{x=0} = G(s)$$

That is

$$U(x,s) = c_1(s)e^{\sqrt{\frac{s}{D}}x} + c_2(s)e^{-\sqrt{\frac{s}{D}}x}$$

Since $u(x,t) < M$ are bounded, $U(x,s) < \frac{M}{s}$ still holds when $x \to \infty$, thus $c_1(s) = 0$

$$U(x,s)|_{x=0} = c_2(s) = G(s) \Rightarrow U(x,s) = G(s)e^{-\sqrt{\frac{s}{D}}x}$$

Look up the table

$$\mathcal{L}^{-1}[\sqrt{\pi}\exp(-a\sqrt{s})] = \frac{a}{2t^{3/2}}\exp\left(\frac{-a^2}{4t}\right), \text{where } a = \frac{x}{\sqrt{D}}$$

We have that

$$K(x,t) \equiv \mathcal{L}^{-1}[e^{-\sqrt{\frac{s}{D}}x}] = \frac{x}{2\sqrt{\pi D}t^{3/2}}\exp\left(\frac{-x^2}{4Dt}\right)$$

In the end

$$u(x,t) = \mathcal{L}^{-1}[G(s)e^{-\sqrt{\frac{s}{D}}x}] = \mathcal{L}^{-1}[G(s)] * \mathcal{L}^{-1}[e^{-\sqrt{\frac{s}{D}}x}] = g(t) * K(x,t) = \int_0^t K(x,t-\tau)g(\tau)d\tau$$

Problem 11

11. (6.5.1 on Pages 422–423)
Solve $u_{tt} = c^2 u_{xx}$ on $t > 0, x > 0$, subject to $u(x,0) = u_t(x,0) = 0$ and boundary condition $u(0,t) = g(t)$

solution
Do the Laplace transform for t

$$\mathcal{L}[u(x,t)] = U(x,s),\ \mathcal{L}[\frac{\partial^2 u(x,t)}{\partial t^2}] = s^2U(x,s) - su(x,t)|_{t=0} - \frac{\partial u(x,t)}{\partial t}|_{t=0} = s^2U(x,s)$$

$$\mathcal{L}[\frac{\partial^2 u(x,t)}{\partial x^2}] = \frac{\partial^2 U(x,s)}{\partial x^2},\ \mathcal{L}[u(x,t)|_{x=0}] = U(x,s)|_{x=0} = G(s)$$

It leads to

$$\frac{\partial^2 U(x,s)}{\partial x^2} = \frac{s^2}{c^2}U(x,s)$$
$$U(x,s)|_{x=0} = G(s)$$

That is

$$U(x,s) = c_1(s)e^{\frac{s}{c}x} + c_2(s)e^{-\frac{s}{c}x}$$

Since $u(x,t) < M$ are bounded, $U(x,s) < \frac{M}{s}$ still holds when $x \to \infty$, thus $c_1(s) = 0$

$$U(x,s)|_{x=0} = c_2(s) = G(s) \Rightarrow U(x,s) = G(s)e^{-\frac{s}{c}x}$$

Here look up the table

$$\mathcal{L}^{-1}[e^{-as}] = \delta(t-a), \quad \text{where } a = \frac{s}{c}$$

Thus

$$u(x,t) = \mathcal{L}^{-1}[G(s)] * \mathcal{L}^{-1}[e^{-\frac{s}{c}x}] = g(t) * \delta(t - \frac{s}{c}) = \begin{cases} g(t-\frac{s}{c}) & \text{if } t \geq \frac{s}{c} \\ 0 & \text{if } t < \frac{s}{c} \end{cases}$$

Problem 10

10. (6.5.2 on Page 432, PDF Page 501)
Use Fourier transforms to find the solution to the initial value problem for the advection-diffusion equation

$$u_t - cu_x - u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$
$$u(x,0) = f(x), \quad x \in \mathbb{R}$$

solution
Do Fourier transform to x

$$\mathcal{F}[u(x,t)] = U(\xi,t),\ \mathcal{F}[\frac{\partial u(x,t)}{\partial t}] = \frac{\partial U(\xi,t)}{\partial t}$$

$$\mathcal{F}[\frac{\partial u(x,t)}{\partial x}] = -i\xi U(\xi,t),\ \mathcal{F}[\frac{\partial^2 u(x,t)}{\partial x^2}] = -\xi^2 U(\xi,t),\ \mathcal{F}[u(x,t)|_{t=0}] = U(\xi,t)|_{t=0} = F(\xi)$$

It leads to

$$\frac{\partial U(\xi,t)}{\partial t} = -ic\xi U(\xi,t) - \xi^2 U(\xi,t)$$

That is

$$U(\xi,t) = A(\xi)e^{-(ic\xi+\xi^2)t}, \quad \text{where } A(\xi) \neq 0$$

By comparing the condition

$$U(\xi,t)|_{t=0} = A(\xi) = F(\xi) \Rightarrow U(\xi,t) = F(\xi)e^{-(ic\xi+\xi^2)t} = \left[F(\xi)e^{-ic\xi t}\right] \cdot e^{-\xi^2 t}$$

Look up the table

$$\mathcal{F}^{-1}[F(\xi)e^{-ic\xi t}] = f(x) * \delta(x-ct) = f(x-ct)$$

$$\mathcal{F}^{-1}\left[\sqrt{\frac{\pi}{a}}e^{-\xi^2/4a}\right] = e^{-ax^2} \quad \text{where } \frac{1}{4a} = t \Leftrightarrow \frac{1}{4t} = a$$

$$\mathcal{F}^{-1}\left[e^{-t\xi^2}\right] = \sqrt{\frac{1}{4\pi t}}e^{-\frac{x^2}{4t}}$$

In the end

$$u(x,t) = \mathcal{F}^{-1}[F(\xi)e^{-ic\xi t}] * \mathcal{F}^{-1}\left[e^{-t\xi^2}\right] = f(x-ct) * \sqrt{\frac{1}{4\pi t}}e^{-\frac{x^2}{4t}} = \sqrt{\frac{1}{4\pi t}}\int_{-\infty}^{\infty} f(y-ct)e^{-\frac{(x-y)^2}{4t}}dy$$

Problem 11

11. (6.5.2 on Page 432, PDF Page 501)
Solve the Cauchy problem for the nonhomogeneous heat equation:

$$u_t = u_{xx} + F(x,t), \quad x \in \mathbb{R}, t > 0; \quad u(x,0) = 0, x \in \mathbb{R}$$

solution
Do Fourier transform to x

$$\mathcal{F}[u(x,t)] = U(\xi,t),\ \mathcal{F}[\frac{\partial u(x,t)}{\partial t}] = \frac{\partial U(\xi,t)}{\partial t},\ \mathcal{F}[F(x,t)] = F(\xi,t)$$

$$\mathcal{F}[\frac{\partial^2 u(x,t)}{\partial x^2}] = -\xi^2 U(\xi,t),\ \mathcal{F}[u(x,t)|_{t=0}] = U(\xi,t)|_{t=0} = 0$$

It leads to

$$\frac{\partial U(\xi,t)}{\partial t} = -\xi^2 U(\xi,t) + F(\xi,t)$$

That is

$$U(\xi,t) = A(\xi)e^{-\xi^2 t} + \int_0^t F(\xi,\tau)e^{-\xi^2(t-\tau)}d\tau, \quad \text{where } A(\xi) \neq 0$$

By comparing the condition

$$U(\xi,t)|_{t=0} = A(\xi) = 0 \Rightarrow U(\xi,t) = \int_0^t F(\xi,\tau)e^{-\xi^2(t-\tau)}d\tau$$

Look up the table

$$\mathcal{F}^{-1}\left[\sqrt{\frac{\pi}{a}}e^{-\xi^2/4a}\right] = e^{-ax^2} \quad \text{where } \frac{1}{4a} = (t-\tau) \Leftrightarrow \frac{1}{4(t-\tau)} = a$$

$$\mathcal{F}^{-1}[e^{-\xi^2(t-\tau)}] = \sqrt{\frac{1}{4\pi(t-\tau)}}e^{-\frac{x^2}{4(t-\tau)}}$$

In the end

$$\begin{aligned} u(x,t) &= \int_0^t \mathcal{F}^{-1}[F(\xi,\tau)] * \mathcal{F}^{-1}[e^{-\xi^2(t-\tau)}]d\tau \\ &= \int_0^t F(x,\tau) * \sqrt{\frac{1}{4\pi(t-\tau)}}e^{-\frac{x^2}{4(t-\tau)}}d\tau \\ &= \int_0^t \int_{-\infty}^{\infty} F(y,\tau)\sqrt{\frac{1}{4\pi(t-\tau)}}e^{-\frac{(x-y)^2}{4(t-\tau)}}dyd\tau \end{aligned}$$

Bonus

(Age of the Earth) In this problem we use Lord Kelvin's argument, given in the mid 1860s, to estimate the age of the earth using a measurement of the geothermal gradient. The geothermal gradient is the derivative of temperature $T_x(0,t)$ measured at the surface of the earth. Since the earth is cooling,the temperature gradient is also decreasing with time. Lord Kelvin's idea of estimating the age of the earth t^* is to treat the earth as flat with $x > 0$ measuring the depth from the surface $x = 0$; assume that the temperature on the surface of the earth was always $0°C$; and solve the heat conduction problem for the temperature of the earth

$$\begin{array}{lll} \text{PDE} & C\rho T_t - KT_{xx} = 0 & \text{for } x > 0, t > 0 \\ \text{BC} & T(0,t) = 0 & \text{for } t > 0 \\ \text{IC} & T(x,0) = T_0 & \text{for } x > 0 \end{array}$$

After finding the solution, we can find the time t^* at which $T_x(0,t)|_{t=t^*}$ equals the current geothermal gradient value of $0.037°C/m$

(a) Solve the initial-boundary value problem.
(b) Assume that $K/C\rho \approx 1.2 \times 10^{-6} m^2/s$ and the initial temperature of the earth was $T_0 \approx 2000°C$(molten rock). Estimate the age of the earth t^* by solving $T_x(x,t)|_{x=0,t=t^*} = 0.037°C/m$. Give your answer in millions of years.

solution

Do the Laplace transform for variable t

$$\mathcal{L}[T(x,t)] = U(x,s),\ \mathcal{L}[\frac{\partial T}{\partial t}] = sU(x,s) - T(x,t)|_{t=0} = sU(x,s) - T_0,$$

$$\mathcal{L}[\frac{\partial^2 T}{\partial x^2}] = \frac{\partial^2 U(x,s)}{\partial x^2},\ \mathcal{L}[T(x,t)|_{x=0}] = U(x,s)|_{x=0} = 0$$

Define $D = \frac{K}{C\rho}$, and it leads to

$$\frac{\partial^2 U(x,s)}{\partial x^2} = \frac{s}{D}U(x,s) - \frac{T_0}{D},\ U(x,s)|_{x=0} = 0$$

We can write $U(x,s) = U_h + U_p$, where the particular solution $U_p = \frac{T_0}{s}$, the homogeneous solution U_h is given by

$$U_h = c_1(s)e^{\sqrt{\frac{s}{D}}x} + c_2(s)e^{-\sqrt{\frac{s}{D}}x}$$

Since $T(x,t) < M$ are bounded, $U = U_h + U_p < \frac{M}{s}$ still holds when $x \to \infty$, thus $c_1(s) = 0$

$$U(x,s) = U_p + U_h = \frac{T_0}{s} + c_2(s)e^{-\sqrt{\frac{s}{D}}x}$$

Compare with the condition

$$U(x,s)|_{x=0} = \frac{T_0}{s} + c_2(s) \cdot 1 = 0 \Rightarrow c_2(s) = -\frac{T_0}{s}$$

Thus, we have

$$U(x,s) = T_0 \left[\frac{1}{s} - \frac{e^{-\sqrt{\frac{s}{D}}x}}{s} \right]$$

Then look up table of Laplace transform

$$\mathcal{L}^{-1}[\frac{e^{-a'\sqrt{s}}}{s}] = 1 - \text{erf}\left(\frac{a'}{2\sqrt{t}}\right) \quad , a' = \frac{x}{\sqrt{D}}, \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-z'^2} dz'$$

Thus

$$T(x,t) = \mathcal{L}^{-1}[U(x,s)] = T_0 \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right) = T_0 \cdot \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-z'^2} dz'$$

For the gradient of x at $x = 0, t = t^*$

$$T_x(x,t)|_{x=0,t=t^*} = T_0 \cdot \frac{2}{\sqrt{\pi}} e^{-\left(\frac{x}{2\sqrt{Dt}}\right)^2} \cdot \frac{1}{2\sqrt{Dt}}|_{x=0,t=t^*} = T_0 \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{Dt^*}} = \frac{T_0}{\sqrt{\pi D t^*}} = 0.037°C/m$$

In the end, the age of earth t^* is

$$t^* = \frac{T_0^2}{0.037^2 \cdot \pi D} \approx \frac{2000^2}{0.037^2 \pi \times 1.2 \times 10^6} \approx 7.750423 \times 10^{14} (s) = 24576430 \text{ (year)} \approx 25 \text{(milion year)}$$

Journal.

Compare and contrast the Laplace transform and the Fourier transform. In particular, discuss when to use which method.

solution

When we apply the Fourier transform, we have to make sure $\int_{-\infty}^{\infty} |f(t)| dt < M$, or other special scenarios like $\sin(t), \cos(t)$

When we applied Laplace Transform, we have to make sure $|f(t)| < Me^{ct}$

We can use Fourier transform when the range of t is $(-\infty, +\infty)$. We can use Laplace transform when the range of t is $(0^{-}, \infty)$