Homework 10 Zhankun Luo

Due: Dec 02, 2020

Problem 1

a. $\begin{cases} u_t = k u_{xx} & 0<x<l,\ t>0 \\ u_x(0,t) = u_x(l,t) = 0, & t>0 \\ u(x,0) = f(x) & 0<x<l \end{cases}$

Solution: Set $u(x,t) = \Sigma\, c_n \cdot X_n(x)\, T_n(t)$

$$\Rightarrow \frac{T_n'}{kT_n} = \frac{X_n''}{X_n} = -\lambda_n$$

BC: $X_n'(0) = X_n'(l) = 0$

$$\Rightarrow \begin{cases} T_n' = -\lambda_n k T_n \Rightarrow T_n = e^{-\lambda_n k t} \\ \begin{cases} X_n'' = -\lambda_n X_n \\ X_n'(0) = X_n'(l) = 0 \end{cases} \Rightarrow \begin{matrix} X_n = \cos(\sqrt{\lambda_n}\, x) \\ \sqrt{\lambda_n}\, l = n\pi \end{matrix} \Rightarrow \begin{matrix} X_n = \cos\left(\frac{n\pi}{l} x\right) \\ \lambda_n = \frac{n^2\pi^2}{l^2} \quad n = 0, 1, 2, \dots \quad n \in Z^* \end{matrix} \end{cases}$$

Then, $u(x,t) = \sum_{n=0}^{+\infty} c_n \cdot X_n(x)\, T_n(t)$

$$= \sum_{n=0}^{+\infty} c_n \cdot \cos\left(\frac{n\pi}{l} x\right) e^{-k\frac{n^2\pi^2}{l^2}t}$$

IC: $u(x,0) = \sum_{n=0}^{+\infty} c_n \cos\left(\frac{n\pi}{l} x\right) = f(x)$

For $n=0$

$$c_0 = \frac{\int_0^l f(x)\cdot 1\, dx}{\int_0^l 1 \cdot 1\, dx} = \frac{1}{l}\int_0^l f(x)\, dx$$

For $n>0$

$$c_n = \frac{\int_0^l f(x)\cos\left(\frac{n\pi}{l} x\right) dx}{\int_0^l \cos^2\left(\frac{n\pi}{l}\right) dx} = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi}{l} x\right) dx$$

To sum up: $u(x,t) = \frac{1}{l}\int_0^l f(x)\, dx + \sum_{n=1}^{+\infty}\left[\frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi}{l} x\right) dx\right]\cos\left(\frac{n\pi}{l} x\right) e^{-k\frac{n^2\pi^2}{l^2}t}$

b. $\begin{cases} u_{tt} = c^2 u_{xx} - a^2 u & 0<x<l,\ t>0 \\ u(0,t) = u(l,t) = 0 & t>0 \\ u(x,0) = f(x) \quad u_t(x,0) = 0 & 0<x<l \end{cases}$

Solution: Set $u(x,t) = \sum C_n \cdot X_n(x) T_n(t)$

$$\Rightarrow \frac{T_n''}{c^2 T_n} + \frac{a^2}{c^2} = \frac{X_n''}{X_n} = -\lambda_n$$

BC: $X_n(0) = X_n(l) = 0$

$$\Rightarrow \begin{cases} T_n'' = -(\lambda_n c^2 + a^2) T_n \Rightarrow T_n = C_{n1} \sin\left(\sqrt{\lambda_n c^2 + a^2}\, t\right) + C_{n2} \cos\left(\sqrt{\lambda_n c^2 + a^2}\, t\right) \\ \begin{cases} X_n'' = -\lambda_n X_n \\ X_n(0) = X_n(l) = 0 \end{cases} \Rightarrow \begin{matrix} X_n = \sin(\sqrt{\lambda_n}\, x) \\ \sqrt{\lambda_n}\, l = n\pi \end{matrix} \Rightarrow \begin{matrix} X_n = \sin\left(\frac{n\pi}{l} x\right) \\ \lambda_n = \frac{n^2\pi^2}{l^2} \quad n = 1, 2, \cdots \\ n \in Z^+ \end{matrix} \end{cases}$$

Then, $u_t(x,0) = \sum_{n=1}^{+\infty} C_n X_n T_n = 0$

$$\Rightarrow T_n = \cos\left(\sqrt{\frac{n^2\pi^2}{l^2} c^2 + a^2}\, t\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{l} x\right) \cdot 1 = f(x)$$

$$C_n = \frac{\int_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) dx}{\int_0^l \sin^2\left(\frac{n\pi}{l} x\right) dx} = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) dx$$

To sum up:

$$u(x,t) = \sum_{n=1}^{+\infty} \left[\frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) dx\right] \sin\left(\frac{n\pi}{l} x\right) \cos\left(\sqrt{\frac{n^2\pi^2}{l^2} c^2 + a^2}\, t\right)$$

Problem 2

Transform the problem

$$\begin{cases} u_t = u_{xx} & 0<x<\pi,\ t>0 \\ u(0,t)=3\ , & u(\pi,t)=1 \\ u(x,0)=f(x) \end{cases}$$

into one with homogeneous boundary conditions

Solution: Let $u(x,t) = v(x,t) + (c_1 x + c_2)$

When $\begin{cases} [c_1 x + c_2]\big|_{x=0} = 3 \\ [c_1 x + c_2]\big|_{x=\pi} = 1 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{-2}{\pi} \\ c_2 = 3 \end{cases}$

Then $v_t = u_t\ ,\ v_{xx} = u_{xx} \Rightarrow v_t = v_{xx}$

BC: $v(0,t) = u(0,t) - [c_1x+c_2]\big|_{x=0} = 3-3 = 0$

$v(\pi,t) = u(\pi,t) - [c_1x+c_2]\big|_{x=0} = 1-1 = 0$

IC: $v(x,0) = u(x,0) - [c_1x+c_2]\big|_{t=0} = f(x) - [c_1x+c_2]$

$= f(x) + \frac{2}{\pi}x - 3$

To sum up: $v(x,t) = u(x,t) - [c_1x+c_2]$

$= u(x,t) + \frac{2}{\pi}x - 3$

$$\begin{cases} v_t = v_{xx} & 0<x<\pi,\ t>0 \\ v(0,t)=0, & u(\pi,t)=0 \\ v(x,0) = f(x)+\frac{2}{\pi}x-3 \end{cases}$$

Problem 3.

Transform the problem with a source term

$$\begin{cases} u_t = u_{xx} + \sin(\frac{x}{2}) \\ u(0,t) = 0 \quad u(\pi,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

into a nonhomogeneous problem, Solve the problem when $f(x) = 1$

Solution:

Set $u(x,t) = v(x,t) + w(x)$

where $\begin{cases} \frac{d^2 w}{dx^2} + \sin(\frac{x}{2}) = 0 \\ w(0) = w(\pi) = 0 \end{cases} \Rightarrow w(x) = -\frac{1}{4}\sin\frac{x}{2} + \frac{1}{4\pi}x$

Then for $v(x,t) \equiv u(x,t) - w(x)$

$$\begin{cases} v_t = v_{xx} \\ v(0,t) = 0, \ v(\pi,t) = 0 \\ v(x,0) = f(x) - w(x) = f(x) + \frac{1}{4}\sin(\frac{x}{2}) - \frac{1}{4\pi}x \end{cases}$$

When $f(x) = 1$

$$\begin{cases} v_t = v_{xx} \\ v(0,t) = 0, \ v(\pi,t) = 0 \\ v(x,0) = \frac{1}{4}\sin(\frac{x}{2}) - \frac{1}{4\pi}x + 1 \end{cases}$$

Set $v = \sum C_n X_n(x) T_n(t)$

$$\Rightarrow \frac{T_n'}{T_n} = \frac{X_n''}{X_n} = -\lambda_n$$

$$\Rightarrow \begin{cases} T_n' = -\lambda_n T_n & \Rightarrow T_n = e^{-\lambda_n t} \\ \begin{cases} X_n'' = -\lambda_n X_n \\ X_n(0) = X_n(\pi) = 0 \end{cases} & \Rightarrow X_n = \sin(\sqrt{\lambda_n} x), \quad \sqrt{\lambda_n}\pi = n\pi \Rightarrow X_n = \sin(n\pi x) \quad \lambda_n = n^2\pi^2 \quad n = 1, 2, \ldots \quad n \in \mathbb{Z}^+ \end{cases}$$

$$\therefore v = \sum_{n=1}^{+\infty} C_n \cdot \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

$$C_n = \frac{\int_0^{\pi} \left[\frac{1}{4}\sin(\frac{x}{2}) - \frac{1}{4\pi}x + 1\right] \sin(n\pi x)\, dx}{\int_0^{\pi} \sin^2(n\pi x)\, dx}$$

$$= \frac{2}{\pi} \int_0^{\pi} \left[\frac{1}{4}\sin(\frac{x}{2}) - \frac{1}{4\pi}x + 1\right] \sin(n\pi x)\, dx$$

$$= \frac{2}{\pi} \left[\frac{\left(\frac{4\pi^4 n^3}{1-4\pi^2 n^2} - 3\pi^2 n\right)\cos(\pi^2 n) + 4\pi^2 n - \sin(\pi^2 n)}{4\pi^3 n^2} \right]$$

$$= \frac{\left(\frac{4\pi^4 n^3}{1-4\pi^2 n^2} - 3\pi^2 n\right)\cos(\pi^2 n) + 4\pi^2 n - \sin(\pi^2 n)}{2\pi^4 n^2}$$

$$\therefore u(x,t) = v + w(x)$$

$$= \sum_{n=1}^{+\infty} C_n \cdot \sin(n\pi x)\, e^{-n^2\pi^2 t} + \left(-\frac{1}{4}\sin\left(\frac{x}{2}\right) + \frac{1}{4\pi}x\right)$$

Problem 6 Bonus

Diffusion constan : D

growth rate : r

length : l

$$\begin{cases} u_t = ru + Du_{xx} \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Solution: Set $u(x,t) = \sum C_n X_n(x) T_n(t)$

$$\Rightarrow \frac{T_n'}{DT_n} - \frac{r}{D} = \frac{X_n''}{X_n} = -\lambda_n$$

BC: $X_n(0) = X_n(l) = 0$

$$\Rightarrow \begin{cases} T_n' = -(D\lambda_n - r) T_n \Rightarrow T_n = e^{-(D\lambda_n - r)t} \\ \begin{cases} X_n'' = -\lambda_n X_n \\ X_n(0) = X_n(l) = 0 \end{cases} \Rightarrow \begin{cases} X_n = \sin(\sqrt{\lambda_n} x) \\ \sqrt{\lambda_n} l = n\pi \end{cases} \Rightarrow X_n = \sin\left(\frac{n\pi}{l} x\right) \quad \lambda_n = \frac{n^2\pi^2}{l^2} \quad n = 1, 2 \ldots \; n \in \mathbb{Z}^+ \end{cases}$$

Then
$$u(x,0) = \sum_{n=1}^{+\infty} C_n \cdot X_n \cdot 1 = f(x)$$
$$= \sum_{n=1}^{+\infty} C_n \sin\left(\frac{n^2\pi^2}{l^2} x\right)$$

$$C_n = \frac{\int_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) dx}{\int_0^l \sin^2\left(\frac{n\pi}{l} x\right) dx} = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) dx$$

Thus:
$$u(x,t) = \sum_{n=1}^{+\infty} \left[\frac{2}{l} \int_0^l f(\xi) \sin\left(\frac{n\pi}{l} \xi\right) d\xi \right] \sin\left(\frac{n\pi}{l} x\right) e^{-\left[D\frac{n^2\pi^2}{l^2} - r\right] t}$$
$$= \int_0^l G(x, \xi, t) f(\xi) d\xi$$

$$G(x, \xi, t) = \frac{2}{l} \sum_{n=1}^{+\infty} e^{-\left[D\frac{n^2\pi^2}{l^2} - r\right] t} \sin\left(\frac{n\pi}{l} \xi\right) \sin\left(\frac{n\pi}{l} x\right)$$

We know that

$$\left|u(x,t)\right| = \left|\int_0^l G(x,\xi,t) f(\xi)\, d\xi\right|$$

$$\leq \int_0^l \left|G(x,\xi,t)\right| \cdot \left|f(\xi)\right| d\xi$$

$$\leq \left|\frac{2}{l}\sum_{n=1}^{+\infty} e^{-\left[D\frac{n^2\pi^2}{l^2}-r\right]t}\right| \cdot \int_0^l \left|f(\xi)\right| d\xi$$

When. $l < \pi\sqrt{\frac{D}{r}} \Rightarrow 1-\frac{rl^2}{D\pi^2} > 0$

$$\Rightarrow D\frac{n^2\pi^2}{l^2}-r = D\frac{\pi^2}{l^2}(n^2-1) + \frac{D\pi^2}{l^2}\left(1-\frac{rl^2}{D\pi^2}\right)$$

$$\geq D\frac{\pi^2}{l^2}\left[3(n-1)\right] + \frac{D\pi^2}{l^2}\left(1-\frac{rl^2}{D\pi^2}\right).$$

$$\left|\frac{2}{l}\sum_{n=1}^{+\infty} e^{-\left[D\frac{n^2\pi^2}{l^2}-r\right]t}\right| \leq \frac{2}{l} e^{-\frac{D\pi^2}{l^2}\left(1-\frac{rl^2}{D\pi^2}\right)t} \sum_{n=1}^{+\infty} e^{-D\frac{\pi^2}{l^2}\left[3(n-1)\right]t}$$

$$= \frac{2}{l} \cdot \frac{1}{1-e^{-D\frac{\pi^2}{l^2}3t}} e^{-\frac{D\pi^2}{l^2}\left(1-\frac{rl^2}{D\pi^2}\right)t}$$

When $t\to\infty$.

$$\frac{1}{1-e^{-D\frac{\pi^2}{l^2}3t}} \to 1, \quad e^{-\frac{D\pi^2}{l^2}\left(1-\frac{r^2l^2}{D\pi^2}\right)t} \to 0$$

$$\Rightarrow \lim_{t\to+\infty}\left|u(x,t)\right| = \lim_{t\to+\infty}\left|\frac{2}{l}\sum_{n=1}^{+\infty} e^{-\left[D\frac{n^2\pi^2}{l^2}-r\right]t}\right| \cdot \int_0^l \left|f(\xi)\right| d\xi = 0$$

Journal.
Compare and contrast the “separation of variables” method for the ordinary differential equations (Section 1.3) and for the partial differential equations (Section 6.4). Discuss what the common ideas behind the two methods, and what the special features of the equations are for the idea to work.

Solution:
The common ideas behind the two methods are converting the original problem to the solvable problem. For the separation of variables in ODE, the ODE equation is converted to the problem of finding integrals on both sides of equations. For the separation of variables in PDE, the PDE solution is written as the weighted sum of fundamental solutions, i.e. the products of X(x) and T(t). We convert the PDE equation to multiple ODE equations.
The special feature of “separation of variables” in the PDE equations is that the PDE equations have to be linear equations.