Problem 2

2. Find the general solution of the following partial differential equations in terms of arbitrary functions

a) $u_{xx} + u = 6y,$ where $u = u(x, y)$
d) $u_{tx} + u_x = 1,$ where $u = u(x, t)$
e) $uu_t = x - t,$ where $u = u(x, t)$. (Hint: $(u^2)_t = 2uu_t$)

(Section 6.1, Problems 2ade on Page 374, PDF Page 433)

solution
a) $u_{xx} + u = 6y,$ where $u = u(x, y)$
treat y as a constant value, the characteristic equation is

$$r^2 + 1 = 0, \quad r_1 = i, r_2 = -i$$

$$u = u_p + u_h, \quad u_p = 6y, \quad u_h = c_1(y)\cos(x) + c_2(y)\sin(x)$$

thus

$$u = 6y + c_1(y)\cos(x) + c_2(y)\sin(x)$$

d) $u_{tx} + u_x = 1,$ where $u = u(x, t)$
set $v \equiv \frac{\partial}{\partial x}u$, it becomes

$$\frac{\partial}{\partial t}v + v = 1$$

multiply the integrating factor $\mu(t) = e^{\int 1 dt} = e^t$

$$\frac{\partial}{\partial t}\left[e^t v\right] = e^t$$

integrate the equation on both sides

$$\frac{\partial}{\partial x}u = v = e^{-t}\left[\int e^t dt + c_1(x)\right] = 1 + c_1'(x)e^{-t}$$

integrate the equation on both sides

$$u = \int 1 + c_1'(x)e^{-t}dx = x + e^{-t}\int c_1'(x)dx + c_2(t) = x + c_1(x)e^{-t} + c_2(t)$$

e) $uu_t = x - t,$ where $u = u(x, t)$. (Hint: $(u^2)_t = 2uu_t$) with $(u^2)_t = 2uu_t$

$$\left(u^2\right)_t = 2(x - t)$$

integrate on both sides, treat x as a constant value

$$u^2 = \int 2(x - t)dt = 2xt - t^2 + c_1(x)$$

thus

$$u = \pm\sqrt{2xt - t^2 + c_1(x)}$$

Problem 3

3. Find a formula for the solution to the (Section 6.1, Problems 3 on Page 374, PDF Page 433)

$$u_{xt} = f(x,t), \quad x,t > 0$$

that satisfies the auxiliary conditions $u(x,0) = g(x), x > 0$ and $u(0,t) = h(t), t > 0$, where f, g, and h are given, well-behaved functions with $g(0) = h(0), g'(0) = h'(0)$

solution

$$\frac{\partial^2}{\partial x' \partial t'} u(x',t') = f(x',t')$$

integrate t' from 0 to x

$$\frac{\partial}{\partial x'} u(x',t) - \frac{\partial}{\partial x'} u(x',t')|_{t'=0} = \int_0^t \frac{\partial^2}{\partial x' \partial t'} u(x',t') dt' = \int_0^t f(x',t') dt'$$

define $\frac{\partial}{\partial x'} u(x',t')|_{t'=0}|_{t'=0} \equiv c_1'(x')$

$$\frac{\partial}{\partial x'} u(x',t) = \int_0^t f(x',t') dt' + c_1'(x')$$

integrate x' from 0 to x

$$u(x,t) - h(t) = u(x,t) - u(0,t) = \int_0^x \frac{\partial}{\partial x'} u(x',t) dx' = \int_0^x \left[\int_0^t f(x',t') dt' \right] dx' + c_1(x)$$

set $t = 0$, notice that $g(0) = h(0)$

$$g(x) - g(0) = u(x,0) - h(0) = \int_0^x 0 dx' + c_1(x) = c_1(x) \Rightarrow c_1(x) = g(x) - g(0)$$

thus

$$u(x,t) = \int_0^x \left[\int_0^t f(x',t') dt' \right] dx' + h(t) + g(x) - g(0)$$

Problem 4

4. Find the general solution of the first-order linear equation(Section 6.1, Problems 4 on Page 374, PDF Page 433)

$$u_t + cu_x = 0$$

by changing variables to the new spatial coordinate $z = x - ct$, where c is a constant (Hint: take $\tau = t, z = x - ct$)

solution

let's guess there is a linear correspondence $\tau = a_1 t + a_2 x, z = b_1 t + b_2 x$

$$\begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \end{pmatrix} = J^T \begin{pmatrix} \frac{\partial}{\partial \tau} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \left[\frac{\partial(\tau,z)}{\partial t}\right]^T \\ \left[\frac{\partial(\tau,z)}{\partial x}\right]^T \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \tau} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \tau} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

here the equation

$$[1 \quad c] \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \end{pmatrix} u = [1 \quad c] \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \tau} \\ \frac{\partial}{\partial z} \end{pmatrix} u = 0$$

if we can set

$$[1 \quad c] \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = [1 \quad 0] \Rightarrow [1 \quad 0] \begin{pmatrix} \frac{\partial}{\partial \tau} \\ \frac{\partial}{\partial z} \end{pmatrix} u = \frac{\partial}{\partial \tau} u = 0$$

here we choose $[a_1, a_2] = [1\ 0], [b_1, b_2] = [-c\ 1]$

$$[1 \quad c] \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} = [1 \quad 0], \quad \tau = a_1 t + a_2 x = t, z = b_1 t + b_2 x = -ct + x$$

integrate $\frac{\partial}{\partial \tau} u = 0$ on both sides

$$u(\tau, z) = \int \frac{\partial}{\partial \tau} u d\tau = \int 0 d\tau = 0 + C(z) = C(z)$$

thus

$$u(t, x) = u(\tau, z) = C(z) = C(-ct + x)$$

Problem 4

4. A homogeneous (constant ρ, C, and K) metal rod has cross-sectional area $A(x), 0 < x < l$, and there is only a small variation of $A(x)$ with x, so that the assumption of constant temperature in any cross section remains valid. There are no sources and the flux is given by $-Ku_x(x,t)$. From a conservation law obtain a partial differential equation for the temperature $u(x,t)$ that reflects the area variation of the bar (Section 6.2, Problems 4 on Page 395)

solution

List the basic laws for the flux J, and the relationship of energy v and temperature u

$$J(x,t) = -K\frac{\partial u}{\partial x}, \quad \mathrm{d}v = C\rho \mathrm{d}u \Leftrightarrow v = C\rho u + \mathrm{const}, \quad f = 0$$

Consider the conservation law, between arbitrary interval $x \in [a,b]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b v(x,t)A(x)\mathrm{d}x = J(a,t)A(a) - J(b,t)A(b)$$

With Leibniz integral rule, the left side becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b v(x,t)A(x)\mathrm{d}x = \int_a^b \frac{\partial v(x,t)}{\partial t}A(x)\mathrm{d}x = \int_a^b A(x)\frac{\partial\left[C\rho u(x,t) + \mathrm{const}\right]}{\partial t}\mathrm{d}x = \int_a^b C\rho A(x)\frac{\partial u}{\partial t}\mathrm{d}x$$

With Fundamental theorem of calculus, the right side becomes

$$J(a,t)A(a) - J(b,t)A(b) = -\int_a^b \frac{\partial\left[-K\frac{\partial u}{\partial x}A(x)\right]}{\partial x}dx = \int_a^b K\left[\frac{\partial A(x)}{\partial x}\frac{\partial u}{\partial x} + A(x)\frac{\partial^2 u}{\partial x^2}\right]\mathrm{d}x$$

The left hand side equals the right hand side

$$\int_a^b \left\{C\rho A(x)\frac{\partial u}{\partial t} - K\left[\frac{\mathrm{d}A(x)}{\mathrm{d}x}\frac{\partial u}{\partial x} + A(x)\frac{\partial^2 u}{\partial x^2}\right]\right\}\mathrm{d}x = 0$$

Because it always holds for arbitrary interval $[a,b]$, the function to be integrated must be 0

$$C\rho A(x)\frac{\partial u}{\partial t} - K\left[\frac{\mathrm{d}A(x)}{\mathrm{d}x}\frac{\partial u}{\partial x} + A(x)\frac{\partial^2 u}{\partial x^2}\right] = 0$$

Problem 6

6. A fluid, having density ρ, specific heat C, and conductivity K, flows at a constant velocity V in a cylindrical tube of length L and radius R. The temperature at position x is $T = T(x,t)$, and diffusion of heat is ignored. As it flows, heat is lost through the lateral side at a rate jointly proportional to the area and to the difference between the temperature T_e of the external environment and the temperature $T(x,t)$ of the fluid (Newton's law of cooling). Derive a partial differential differential equation model for the temperature $T(x,t)$. Find the general solution of the equation by transforming to a moving coordinate system $z = x - Vt, \tau = t$
(Section 6.2, Problems 6 on Page 395)

solution
correction:
K should not be the conductivity of diffusion, it should be the **Heat transfer coefficient** of Newton cooling, because we neglect the effect of diffusion in the problem. They are different concepts, thanks to my friend Xiang Li for his explanation, he is a Thermal Engineering graduate student.

List the basic physical laws
here u is heat energy, J is the heat flux from convection, $\bar{J}$ is the heat flux from heat loss

$$\mathrm{d}u = C\rho \mathrm{d}T \Leftrightarrow u = C\rho T + \text{const}, \quad J(x,t) \approx Vu = V\left[C\rho T + \text{const}\right], \quad \bar{J}(x,t) = -K\left[T(x,t) - T_e\right]$$

The energy conservation law, in the arbitrary interval $[a,b]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u \cdot \pi R^2 \mathrm{d}x = \left[J(a,t) - J(b,t)\right] \cdot \pi R^2 + \int_a^b \bar{J}(x,t) \cdot 2\pi R \mathrm{d}x$$

The left hand side, with Leibniz integral rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u \cdot \pi R^2 \mathrm{d}x = \int_a^b \pi R^2 \frac{\partial u}{\partial t} dx = \int_a^b \pi R^2 \frac{\partial\left[C\rho T + \text{const}\right]}{\partial t} dx = \int_a^b \pi R^2 C\rho \left[\frac{\partial T}{\partial t}\right] dx$$

The first term of right hand side, with Fundamental theorem of calculus

$$\left[J(a,t) - J(b,t)\right] \cdot \pi R^2 = -\int_a^b \pi R^2 \frac{\partial J(x,t)}{\partial x} dx = -\int_a^b \pi R^2 \frac{\partial\left(V\left[C\rho T + \text{const}\right]\right)}{\partial x} dx = -\int_a^b \pi R^2 C\rho \left[V\frac{\partial T}{\partial x}\right] dx$$

The second term of right hand side

$$\int_a^b \bar{J}(x,t) \cdot 2\pi R \mathrm{d}x = \int_a^b -K\left[T(x,t) - T_e\right] \cdot 2\pi R \mathrm{d}x = -\int_a^b 2\pi RK\left[T(x,t) - T_e\right] \mathrm{d}x$$

Thus

$$\int_a^b \left\{\pi R^2 C\rho \left[\frac{\partial T}{\partial t}\right] + \pi R^2 C\rho \left[V\frac{\partial T}{\partial x}\right] + 2\pi RK\left[T - T_e\right]\right\} \mathrm{d}x = 0$$

Because $[a,b]$ ia a arbitrary interval, the function to be integrated must be 0

$$\pi R^2 C\rho \left[\frac{\partial T}{\partial t}\right] + \pi R^2 C\rho \left[V\frac{\partial T}{\partial x}\right] + 2\pi RK\left[T - T_e\right] = 0$$

Thus

$$\left[\frac{\partial T}{\partial t} + V\frac{\partial T}{\partial x}\right] + \frac{2K}{RC\rho}\left[T - T_e\right] = 0$$

Consider the transformation $z = x - Vt, \tau = t$

$$J = \frac{\partial(z,\tau)}{\partial(x,t)} = \begin{pmatrix} 1 & -V \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial t} \end{pmatrix} = J^T \begin{pmatrix} \frac{\partial T}{\partial z} \\ \frac{\partial T}{\partial \tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -V & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial z} \\ \frac{\partial T}{\partial \tau} \end{pmatrix}$$

For the first term of equation

$$\left[\frac{\partial T}{\partial t} + V\frac{\partial T}{\partial x}\right] = \begin{pmatrix} V & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial t} \end{pmatrix} = \begin{pmatrix} V & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -V & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial z} \\ \frac{\partial T}{\partial \tau} \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial z} \\ \frac{\partial T}{\partial \tau} \end{pmatrix} = \frac{\partial T(z,\tau)}{\partial \tau}$$

The equation becomes

$$\frac{\partial T(z,\tau)}{\partial \tau} + \frac{2K}{RC\rho}\left[T - T_e\right] = 0$$

(1) When $T - T_e = 0$

$$T(x,t) = T(z,\tau) = T_e$$

(2) When $T - T_e \neq 0$

$$\frac{\partial \ln|T - T_e|}{\partial \tau} = \frac{\mathrm{d}\ln|T - T_e|}{\mathrm{d}\left[T - T_e\right]}\frac{\partial T(z,\tau)}{\partial \tau} = \frac{1}{T - T_e}\frac{\partial T(z,\tau)}{\partial \tau} = -\frac{2K}{RC\rho}$$

Integrate on both sides, here $C(z)$ is an arbitrary function

$$\ln|T - T_e| = \int \frac{\partial \ln|T - T_e|}{\partial \tau}\mathrm{d}\tau = -\int \frac{2K}{RC\rho}\mathrm{d}\tau = -\frac{2K}{RC\rho}\tau + C(z)$$

That is, where $A(z) \neq 0$ for $\forall z$

$$T = A(z)\exp\left(-\frac{2K}{RC\rho}\tau\right) + T_e$$

With th transformation $z = x - Vt, \tau = t$

$$T(x,t) = A(x - Vt)\exp\left(-\frac{2K}{RC\rho}t\right) + T_e$$

BONUS.

If it takes 3 hours to roast a 15 lb turkey, how long will it take to roast a 20 lb one in the same oven? Give your best estimate and explain your method.

solution

Set symbols as follow:
$k \equiv \frac{C\rho}{K}$, where C is Specific heat capacity, ρ is density, K is the heat conductivity
$u(r,t)$ is temperature of turkey, u_e is the temperature of oven, u_0 is the initial temperature of turkey, here we treat the the turkey as a sphere of radius R
h is the Heat transfer coefficient of Newton cooling

Based on the Example 6.13 in the textbook, we can write equation

$$\frac{\partial u}{\partial t} = k\Delta u$$

$$u(r,t)|_{t=0} = u_0, \quad -K\frac{\partial u}{\partial r}|_{r=R} = h(u-u_e)|_{r=R}, \quad \frac{\partial u}{\partial r}|_{r=0} = 0$$

Now, define $v = u - u_e$, and the boundary conditions are symmetric for angle angles, thus $v = v(r,t)$

$$\Delta(\cdot) = \left[\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right]\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right](\cdot)$$

Thus

$$\frac{\partial v}{\partial t} = k\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left[r^2\frac{\partial v}{\partial r}\right]\right)$$

$$v(r,t)|_{t=0} = u_0 - u_e, \quad \left(hv + K\frac{\partial v}{\partial r}\right)|_{r=R} = 0, \quad \frac{\partial v}{\partial r}|_{r=0} = 0$$

Consider to write the function v as the weighted sum of separable $v_n = y_n(r)\phi_n(t)$

$$v(r,t) = \sum c_n y_n(r)\phi_n(t) \Rightarrow \frac{-\frac{1}{r^2}\left(\frac{d}{dr}\left[r^2\frac{dy_n(r)}{dr}\right]\right)}{y_n(r)} = \frac{-\frac{d\phi_n(t)}{dt}}{k\phi_n(t)} = \bar{\lambda}_n$$

It leads to

$$-\frac{1}{r^2}\left(\frac{d}{dr}\left[r^2\frac{dy_n}{dr}\right]\right) = \bar{\lambda}_n y_n$$

$$-\frac{d\phi_n(t)}{dt} = k\bar{\lambda}_n\phi_n(t)$$

Here we can solve $\phi_n(t)$

$$\phi_n(t) = e^{-k\bar{\lambda}_n t}$$

Let's come back to the y_n

$$-\left(\frac{d}{dr}\left[r^2\frac{dy_n}{dr}\right]\right) = \bar{\lambda}_n r^2 y_n$$

$$\left(hy_n + K\frac{dy_n}{dr}\right)|_{r=R} = 0, \quad \frac{dy_n}{dr}|_{r=0} = 0$$

To eliminate R, assume **heat transfer$\gg$heat diffusion**, $h \gg \frac{K}{R}$ at $r = R$. Substitute $x \equiv r/R$,

$$-\left(\frac{d}{dx}\left[x^2\frac{dy_n}{dx}\right]\right) = \bar{\lambda}_n R^2 x^2 y_n = \lambda_n x^2 y_n$$

$$y_n|_{x=1} = 0, \quad \frac{dy_n}{dx}|_{x=0} = 0$$

So, we have the the correspondence, here λ_n is a fixed value for any R

$$\bar{\lambda}_n = \frac{\lambda_n}{R^2}, \quad \phi_n(t) = e^{-k\lambda_n \frac{t}{R^2}}$$

The equation of y_n is a SL problem, here λ_n is the eigenvalue, y_n is the corresponding eigenfunction

$$\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] + q(x), \quad p(x) = x^2, \quad q(x) = 0$$

$$\mathcal{L}y_n = \lambda_n w(x) y_n, \quad w(x) = x^2$$

We can verify that

$$\int_a^b y_n \frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right] dx = \left[y_n p(x)\frac{dy_m}{dx}\right]_a^b - \int_a^b p(x)\frac{dy_n}{dx}\frac{dy_m}{dx} dx \textbf{ symmetric form}$$

$$\int_a^b y_n q(x) y_m dx \textbf{ symmetric form}$$

Thus

$$\int_a^b y_n \mathcal{L} y_m - y_m \mathcal{L} y_n = \left[y_m p(x)\frac{dy_n}{dx} - y_n p(x)\frac{dy_m}{dx}\right]_a^b = \left[p(x)\left(y_m\frac{dy_n}{dx} - y_n\frac{dy_m}{dx}\right)\right]_a^b$$

Set $[a, b] = [0, 1]$, notice $p(0) = 0$, $\frac{dy_n}{dx}|_{x=0} = 0$, $\frac{dy_m}{dx}|_{x=0} = 0$

$$\left[p(x)\left(y_m\frac{dy_n}{dx} - y_n\frac{dy_m}{dx}\right)\right]_0^1 = 0 - 0 = 0$$

$$\int_0^1 y_n \mathcal{L} y_m - y_m \mathcal{L} y_n = \int_0^1 y_n \lambda_m w(x) y_m - y_m \lambda_n w(x) y_n = 0$$

We can define the bracket as

$$\langle f, g\rangle = \int_0^1 f\ w(x)\ g dx, \quad \langle y_n, \lambda_m y_m\rangle = \int_0^1 y_n w(x) \lambda_m y_m dx, \quad \langle \lambda_n y_n, y_m\rangle = \int_0^1 \lambda_n y_n w(x) y_m dx$$

$$\langle \lambda_n y_n, y_m\rangle = \langle y_n, \lambda_m y_m\rangle$$

For $\lambda_n \neq \lambda_m$

$$\langle y_n, y_m\rangle = 0 \Leftrightarrow (\lambda_n - \lambda_m)\langle y_n, y_m\rangle = 0 \Leftrightarrow \langle \lambda_n y_n, y_m\rangle = \langle y_n, \lambda_m y_m\rangle$$

If λ_n has multiple eigenfunctions y_n, y_n', Gram–Schmidt process can make sure $\langle y_n, y_n'\rangle = 0$
Currently, we can calculate c_n, with the boundary condition

$$v(r,t)|_{t=0} = \sum c_n y_n\left(\frac{r}{R}\right)\phi_n(t)|_{t=0} = \sum c_n y_n(x) e^{-k\lambda_n \frac{t}{R^2}}|_{t=0} = \sum c_n y_n(x) = u_0 - u_e$$

$$(u_0 - u_e)\langle 1, y_n\rangle = \langle u_0 - u_e, y_n\rangle = \langle \sum c_n y_n, y_n\rangle = c_n\langle y_n, y_n\rangle$$

Thus

$$c_n = (u_0 - u_e)\frac{\langle 1, y_n\rangle}{\langle y_n, y_n\rangle} = (u_0 - u_e)\frac{\int_0^1 1\ w(x) y_n dx}{\int_0^1 y_n w(x) y_n dx} = (u_0 - u_e)\frac{\int_0^1 x^2 y_n dx}{\int_0^1 x^2 y_n^2 dx}$$

$$u(r,t) = v(r,t) + u_e = (u_0 - u_e)\left[\sum \frac{\int_0^1 x^2 y_n dx}{\int_0^1 x^2 y_n^2 dx} y_n\left(\frac{r}{R}\right) e^{-k\lambda_n \frac{t}{R^2}}\right] + u_e$$

Expand the eigenfunction $y(x) = \sum_{k=0}^{\infty} a_k x^k$, compare the coefficient of x^k

$$-(k+1)ka_k = \lambda a_{k-2} \Rightarrow \frac{a_k}{a_{k-2}} = \frac{(-\lambda)}{(k+1)k}, \quad \frac{dy}{dx}|_{x=0} = a_1 = 0 \Rightarrow a_{2k+1} = 0,\ k \in \mathbb{Z}^*$$

notice that, set $y(0) = a_0 = 1$

$$\frac{a_{2k}}{a_0} = \prod_{l=1}^{k} \frac{a_{2l}}{a_{2l-2}} = \prod_{l=1}^{k} \frac{(-\lambda)}{(l+1)l} = \frac{(-\lambda)^k}{(2k+1)!} \Rightarrow a_{2k} = \frac{(-\lambda)^k}{(2k+1)!}$$

With the boundary condition for eigenfunction $y(1) = 0$

$$y(1) = \sum_{k=0}^{\infty} a_{2k} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(2k+1)!} = \frac{1}{\sqrt{\lambda}} \sum_{k=0}^{\infty} (-1)^k \frac{(\sqrt{\lambda})^{2k+1}}{(2k+1)!} = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = 0$$

Thus

$$\sqrt{\lambda_n} = n\pi \Rightarrow \lambda_n = n^2\pi^2,\ y_n(x) = \sum_{k=0}^{\infty} \frac{(-n^2\pi^2)^k}{(2k+1)!} x^{2k} = \frac{\sin(n\pi x)}{n\pi x}, \quad n \in \mathbb{Z}^+$$

For the coefficientc_n

$$c_n = \frac{\int_0^1 x^2 y_n dx}{\int_0^1 x^2 y_n^2 dx} = \frac{\int_0^1 x^2 \frac{\sin(n\pi x)}{n\pi x} dx}{\int_0^1 x^2 \frac{\sin^2(n\pi x)}{n^2\pi^2 x^2} dx} = \frac{\frac{(-1)^{n+1}}{n^2\pi^2}}{\frac{1}{2n^2\pi^2}} = 2(-1)^{n+1}$$

$$u(r,t) = (u_0 - u_e)\left[\sum_{n=1}^{\infty} 2(-1)^{n+1} \frac{\sin(n\pi \frac{r}{R})}{n\pi \frac{r}{R}} e^{-kn^2\pi^2 \frac{t}{R^2}}\right] + u_e$$

The meaning of the cooked turkey is that the inner temperature must reach the temperature threshold u_{th} at the cooked time t^*. i.e. $u(r,t)|_{r=0,t=t^*} = u_{th}$

$$u(r,t)|_{r=0,t=t^*} = (u_0 - u_e)\left[\sum_{n=1}^{\infty} 2(-1)^{n+1} e^{-kn^2\pi^2 \frac{t^*}{R^2}}\right] + u_e = u_{th}$$

notice that $\lambda_n, y_n, k, u_e, u_0$ won't change when the radius R of turkey changes, it implies

$$\frac{t^*}{R^2} = f\left(\frac{u_{th} - u_e}{u_0 - u_e}\right)/k = \text{const}$$

That is the relationship of cooking time t^* and radius R,
under the assumption: **heat transfer** $\gg$ **heat diffusion**, i.e. i.e. $h \gg \frac{K}{R}$ at $r = R$

$$\frac{R^3}{m} = \frac{3}{4\pi\rho} = \text{const}$$

Finally

$$\frac{t^*}{m^{\frac{2}{3}}} = \frac{t^*}{R^2} \cdot \left[\frac{R^3}{m}\right]^{\frac{2}{3}} = \text{const}$$

It takes t_1^*=3 hours to roast m_1=15 lb turkey, we wan to know how long will it take t_2^* to roast a m_2 =20 lb one in the same oven

$$\frac{t_1^*}{m_1^{\frac{2}{3}}} = \frac{t_2^*}{m_2^{\frac{2}{3}}} = \text{const} \Longrightarrow t_2^* = \left(\frac{m_2}{m_1}\right)^{\frac{2}{3}} t_1^* = \left(\frac{20}{15}\right)^{\frac{2}{3}} \cdot 3 = 3.634241 \approx 3.63$$

To sum up, it take $t_2^* \approx 3.63$ hours to roast a $m_2 = 20$ lb one in the same oven

Journal.

Compare and contrast the derivation of Equation (1.3)(**steady-state heat conduction equation**) in Section 5.1 and of Equation (2.7)(**reaction–diffusion equations**) in Section 6.2

$$-\frac{d}{dx}\left(K(x)\frac{T(x)}{dx}\right) = f(x), \quad 0 < x < L \tag{1.3}$$

$$u_t = Du_{xx} + f(x,t,u) \tag{2.7}$$

Discuss how the conservation law is used in the two cases. Why in one case we obtain an ordinary differential equation, but in the other case we obtain a partial differential equation?
note: come from (Fourier's law), and (Fick's law)
where $K(x)$: **thermal conductivity**, D: **diffusion constant**

$$\phi(x) = -K(x)\frac{T(x)}{dx} \tag{Fourier's law}$$

$$J(x,t) = -Du_x(x,t) \tag{Fick's law}$$

and basic equation (steady-state assumption), and (local form of the conservation law)

$$\phi'(x) = f(x) \tag{steady-state assumption}$$

$$u_t + \nabla \cdot \mathbf{J} = f(\mathbf{x}, t, u) \tag{local form of the conservation law}$$

solution
(1) Derivation of (2.7)(**reaction–diffusion equations**)
In the one dimension scenario, $\mathbf{x}$ becomes x, $\nabla \cdot \mathbf{J}$ becomes $\frac{\partial J(x,t)}{\partial x}$

$$u_t + \frac{\partial J(x,t)}{\partial x} = f(x,t,u)$$

With (Fick's law)

$$u_t + \frac{\partial\left[-Du_x(x,t)\right]}{\partial x} = u_t - D\frac{\partial u_x(x,t)}{\partial x} = u_t - Du_{xx} = f(x,t,u)$$

That is (2.7)(**reaction–diffusion equations**)

$$u_t = Du_{xx} + f(x,t,u)$$

(2) Derivation of (1.3)(**steady-state heat conduction equation**)
For steady-state, $u(x,t) = u(x), u_t = 0, u_x = \frac{du(x)}{dx}$, and $J(x,t) = \phi(x), f(x,t,u) = f(x)$
Notice that

$$du(x) = C(x)\rho(x)dT(x) \Rightarrow \frac{du(x)}{dx} = C(x)\rho(x)\frac{dT(x)}{dx}$$

Thus, with (Fick's law)

$$\phi(x) = J(x,t) = -Du_x(x,t) = -D\frac{du(x)}{dx} = -DC(x)\rho(x)\frac{dT(x)}{dx} = -K(x)\frac{dT(x)}{dx}$$

Here we define the conductivity $K(x) \equiv DC(x)\rho(x)$, that is (Fourier's law)
The conservation law is

$$0 + \frac{d\phi(x)}{dx} = u_t + \frac{\partial J(x,t)}{\partial x} = f(x,t,u) = f(x)$$

To substitute $\phi(x) = -K(x)\frac{dT(x)}{dx}$, that is (1.3)(**steady-state heat conduction equation**)

$$-\frac{d}{dx}\left(K(x)\frac{T(x)}{dx}\right) = f(x)$$