Problem 8

8. Find the eigenvalues λ for the boundary value problem

$$-u'' - 2u' = \lambda u, \quad 0 < x < 1$$
$$u(0) = 0, \quad u(1) = 0$$

(Section 5.1, Problem 8(on Page 284), PDF Page 333)

solution
The characteristic equation

$$-r^2 - 2r = \lambda$$

The roots r_1, r_2 has $r_1 + r_2 = -2, \ r_1 r_2 = \lambda$
(a) $r_1 \neq r_2$
the solution is in form of

$$u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

thus

$$u(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$
$$u(1) = c_1 e^{r_1} + c_2 e^{r_2} = c_1(e^{r_1} - e^{r_2}) = 0 \Rightarrow c_1 = -c_2 = 0$$

Then $u = 0$, it is not eigenfunction, there is no eigenvalue λ
(b) $r_1 = r_2$
thus $r_1 = r_2 = -1$, consequently, $\lambda = r_1 r_2 = 1$ the solution is in form of

$$u = (c_1 + c_2 t)e^{r_1 t} = (c_1 + c_2 t)e^{-t}$$

thus

$$u(0) = c_1 = 0$$
$$u(1) = (c_1 + c_2)e^{-1} = c_2 e^{-1} = 0 \Rightarrow c - 2 = 0$$

Then $u = 0$, it is not eigenfunction, there is no eigenvalue λ
Conclusion: to sum up (a)(b), there is no eigenvalue λ

Problem 1

1. Show that the SLP

$$-y''(x) = \lambda y(x), \quad 0 < x < l$$
$$y(0) = 0, \quad y(l) = 0$$

has eigenvalues $\lambda_n = n^2\pi^2/l^2$ and corresponding eigenfunctions $y_n(x) = \sin(n\pi x/l), n = 1, 2, \cdots$ (Subsection 5.2.2, Problems 1(on Pages 300–301), PDF Page 351)

solution

The characteristic equation

$$-r^2 = \lambda$$

The roots r_1, r_2 has $r_1 + r_2 = 0,\ r_1 r_2 = \lambda$

(a) $r_1 \neq r_2$

the solution is in form of

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

thus, $r_1 = -r_2 \neq 0$

$$y(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$
$$y(l) = c_1 e^{r_1 l} + c_2 e^{r_2 l} = c_1(e^{r_1 l} - e^{-r_1 l})$$

because $u \neq 0 \Rightarrow c_1 \neq 0$, thus

$$e^{r_1 l} - e^{-r_1 l} = 0 \Leftrightarrow e^{2r_1 l} = 1 \Leftrightarrow 2r_1 l = n \cdot 2\pi i \Leftrightarrow r_1 = n\pi i/l,\ n \in \mathbb{Z}, n \neq 0$$

that is

$$r_1 = n\pi i/l,\ r_2 = -n\pi i/l,\ \lambda_n = r_1 r_2 = n^2\pi^2/l^2 \quad n \in \mathbb{Z}^+$$

the eigenfunction is

$$y_n = c_1(e^{r_1 x} - e^{-r_1 x}) = 2c_1 i \sin(n\pi x/l)$$

select $2c_1 i = 1$

$$y_n = \sin(n\pi x/l)$$

(b) $r_1 = r_2$

thus $r_1 = -r_2 = 0, \lambda = r_1 r_2 = 0$

the solution in form of

$$y = c_1 + c_2 t$$

here $y(0) = 0,\ y(l) = 0$, makes $c_1 = c_2 = 0$, thus $y = 0$, so there is no eigenfunction

Conclusion: to sum up (a)(b), eigenvalues $\lambda_n = n^2\pi^2/l^2$ have corresponding eigenfunctions $y_n(x) = \sin(n\pi x/l), n \in \mathbb{Z}^+$

Problem 3

3. Find the eigenvalues and eigenfunctions for the problem with periodic boundary conditions:

$$-y''(x) = \lambda y(x), \quad 0 < x < l$$
$$y(0) = y(l),\ y'(0) = y'(l)$$

(Hint for Problem 3: A system of linearhomogeneous equations has nonzero solutions if and only if the determinant of the coefficient matrix iszero.)
(Subsection 5.2.2, Problems 3(on Pages 300–301), PDF Page 351)

solution
The characteristic equation, roots r_1, r_2 have $r_1 + r_2 = 0,\ r_1 r_2 = \lambda$

$$-r^2 = \lambda$$

(a) $r_1 \neq r_2$
the solution is in form of, here $r_1 = -r_2 \neq 0$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{r_1 x} + c_2 e^{-r_1 x}$$

thus

$$y(0) = c_1 + c_2 = c_1 e^{r_1 l} + c_2 e^{-r_1 l} = y(l) \Rightarrow [e^{r_1 l} - 1 \quad e^{-r_1 l} - 1] \cdot [c_1\ c_2]^T = 0$$
$$y'(0) = c_1 r_1 - c_2 r_1 = c_1 r_1 e^{r_1 l} - c_2 r_1 e^{-r_1 l} = y(l) \Rightarrow [r_1(e^{r_1 l} - 1) \quad - r_1(e^{-r_1 l} - 1)] \cdot [c_1\ c_2]^T = 0$$

combine them

$$\begin{pmatrix} e^{r_1 l} - 1 & e^{-r_1 l} - 1 \\ r_1(e^{r_1 l} - 1) & - r_1(e^{-r_1 l} - 1) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0}$$

because for eigenfunction it must has nonzero function for $[c_1\ c_2]^T$, thus

$$\begin{vmatrix} e^{r_1 l} - 1 & e^{-r_1 l} - 1 \\ r_1(e^{r_1 l} - 1) & - r_1(e^{-r_1 l} - 1) \end{vmatrix} = 2r_1 e^{-r_1 l}(e^{r_1 l} - 1)^2 = 0$$

so that

$$e^{r_1 l} - 1 = 0 \Leftrightarrow r_1 l = n \cdot 2\pi i \Leftrightarrow r_1 = 2n\pi i/l,\ n \in \mathbb{Z}, n \neq 0$$

that is

$$r_1 = 2n\pi i/l,\ r_2 = -2n\pi i/l,\ \lambda_n = r_1 r_2 = 4n^2\pi^2/l^2 \quad n \in \mathbb{Z}^+$$

respectively

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = k_1 \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + k_2 \begin{pmatrix} -(1/2)i \\ (1/2)i \end{pmatrix}$$

when $(k_1, k_2) = (1, 0)$ the eigenfunction is

$$y_n = \cos(2n\pi x/l)$$

when $(k_1, k_2) = (0, 1)$ the eigenfunction is

$$y_n^* = \sin(2n\pi x/l)$$

(b) $r_1 = r_2$
thus $r_1 = -r_2 = 0, \lambda = r_1 r_2 = 0$
the solution in form of

$$y = c_1 + c_2 t$$

$y(0) = y(l),\ y'(0) = y'(l)$, make $c_2 = 0$, thus $y = c_1 \cdot 1$, so there is eigenfunction $y_0 = 1$ for $\lambda_0 = 0$

Conclusion: to sum up (a)(b), eigenvalues $\lambda_n = 4n^2\pi^2/l^2$ have corresponding eigenfunctions $y_n = \cos(2n\pi x/l), y_n^* = \sin(2n\pi x/l) \quad n \in \mathbb{Z}^+$, moreover, for $\lambda_0 = 0$, the eigenfunction $y_0 = 1$

Problem 1

1. Verify that the set of functions $\cos(n\pi x/l), n = 0, 1, 2\cdots$, form an orthogonal set on the interval [0, l]. If (Subsection 5.2.3, Problems 1(on Page 308), PDF Page 360)

$$f(x) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{l}\right)$$

converges in the mean-square sense on [0, l], what are the formula for the c_n? This series is called the **Fourier cosine series** on [0, l]. Find the Fourier cosine series for $f(x) = 1 - x$ on [0, 1]

solution

For the function $f_n = \cos(n\pi x/l)$

$$(f_n, f_n) = \int_0^l f_n \bar{f}_n dx = \int_0^l \cos^2\left(\frac{n\pi x}{l}\right) dx = \int_0^l [1/2 + \cos\left(\frac{2n\pi x}{l}\right)]dx = \begin{cases} \frac{l}{2} & n \neq 0 \\ l & n = 0 \end{cases}$$

For different functions $f_n = \cos(n\pi x/l),\ f_m = \cos(m\pi x/l)$, here $n \neq m$

$$(f_n, f_m) = \int_0^l f_n \bar{f}_m dx = \int_0^l \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right) dx = \frac{1}{2}\int_0^l \cos\left(\frac{(n+m)\pi x}{l}\right) + \cos\left(\frac{(n-m)\pi x}{l}\right) dx = 0$$

(1) from above, we verify the set of functions $\cos(n\pi x/l), n \in Z^*$, form an orthogonal set on [0, l]
(2) consider the formula for the c_n

$$(f, f_n) = (\left[\sum_{m=0}^{\infty} c_m f_m\right], f_n) = \sum_{m=0}^{\infty} c_m (f_m, f_n) = c_n(f_n, f_n) = \begin{cases} c_n \frac{l}{2} & n \neq 0 \\ c_n l & n = 0 \end{cases}$$

that is

$$c_0 = \frac{1}{l}(f, f_0) = \frac{1}{l}\int_0^l f(x)dx$$

$$c_n = \frac{2}{l}(f, f_n) = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right) dx \quad n \neq 0$$

(3) $f(x) = 1 - x$ on [0, 1], here $l = 1$, the function $f_n = \cos(n\pi x)$

$$c_0 = \int_0^1 [1 - x]dx = \frac{1}{2}$$

$$c_n = 2\int_0^1 [1 - x]\cos(n\pi x)\, dx \quad n \neq 0$$

here for $n \neq 0$

$$\int_0^1 \cos(n\pi x)\, dx = \frac{1}{n\pi}\sin(n\pi x)\,|_0^1 = 0$$

$$\int_0^1 x\cos(n\pi x)\, dx = \frac{1}{n\pi}\left[x\sin(n\pi x)\,|_0^1 - \int_0^1 \sin(n\pi x)\, dx\right] = \frac{1}{n\pi}\left[0 + \frac{1}{n\pi}\cos(n\pi x)\,|_0^1\right] = \frac{1}{(n\pi)^2}\left[(-1)^n - 1\right]$$

we conclude

$$c_0 = \frac{1}{2}, \quad c_n = \frac{2}{(n\pi)^2}\left[1 - (-1)^n\right] \quad n \neq 0$$

Fourier cosine series for $f(x) = 1 - x$ on [0, 1]

$$f(x) = \sum_{n=0}^{\infty} c_n f_n = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}\left[1 - (-1)^n\right]\cos(n\pi x)$$

Problem 2

2. Let f be defined and integrable on [0, l]. (Subsection 5.2.3, Problems 2(on Page 308), PDF Page 360) The orthogonal expansion

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

is called the Fourier sine series for f on [0, 1].
Find the Fourier sine series for $f(x) = \cos x$ on $[0, \pi/2]$.
What is the Fourier sine series of $f(x) = \sin x$ on $[0, \pi]$?

solution
(1) Fourier sine series of $f(x) = \cos x$ on $[0, \pi/2]$, here $l = \pi/2$, for $n \in \mathbb{Z}^+$

$$\begin{aligned} b_n &= \frac{4}{\pi}\int_0^{\pi/2} \cos x \cdot \sin(2nx)dx = \frac{2}{\pi}\int_0^{\pi/2} \sin((2n+1)x) + \sin((2n-1)x)dx \\ &= -\frac{2}{\pi}\left[\frac{\cos((2n+1)x)|_0^{\pi/2}}{2n+1} + \frac{\cos((2n-1)x)|_0^{\pi/2}}{2n-1}\right] \\ &= \frac{2}{\pi}\left[\frac{1-0}{2n+1} + \frac{1-0}{2n-1}\right] = \frac{2}{\pi}\frac{4n}{4n^2-1} = \frac{8n}{\pi(4n^2-1)} \end{aligned}$$

thus

$$f(x) = \cos x = \sum_{n=1}^{\infty} \frac{8n}{\pi(4n^2-1)}\sin(2nx)$$

(2) Fourier sine series of $f(x) = \sin x$ on $[0, \pi]$, here $l = \pi$, for $n \in \mathbb{Z}^+$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \sin x \cdot \sin(nx)dx = \frac{1}{\pi}\int_0^{\pi} \cos((n-1)x) - \cos((n+1)x)dx \\ &= \begin{cases} \frac{1}{\pi}\pi = 1 & n = 1 \\ \frac{1}{\pi}\left[\frac{\sin((n-1)x)|_0^{\pi}}{n-1} - \frac{\sin((n+1)x)|_0^{\pi}}{n+1}\right] = 0 & n \neq 1,\ n \in \mathbb{Z}^+ \end{cases} \end{aligned}$$

that is

$$b_n = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1,\ n \in \mathbb{Z}^+ \end{cases}$$

thus

$$\sin x = \sum_{n=1}^{\infty} b_n \sin(nx) = \sin x$$

Bonus.

In class (on Oct. 26) we derived the boundary value problem for the hanging cable

$$y'' = a\sqrt{1+(y')^2}, \quad y(-h) = y(h) = 0$$

where $a = g\rho/K$. Solve this problem and use software to graph the solution with $a = h = 1$ (Problem 1 on Page 165, PDF Page 197)

solution
Substitute $v \equiv y'$

$$\frac{dv}{dx} = a\sqrt{1+v^2}$$

thus

$$\sinh^{-1}(v) = \ln(\sqrt{1+v^2}+v) = \int \frac{dv}{\sqrt{1+v^2}} = a\int dx = ax + c_1$$

so that

$$\frac{dy}{dx} = v = \sinh(ax+c_1) = \frac{e^{ax+c_1} - e^{-(ax+c_1)}}{2}$$

then

$$y = \int \sinh(ax+c_1)dx = \frac{e^{ax+c_1} + e^{-(ax+c_1)}}{2a} + c_0 = \frac{\cosh(ax+c_1)}{a} + c_0$$

with boundary conditions

$$y(h) = \frac{\cosh(ah+c_1)}{a} + c_0 = 0, \quad y(-h) = \frac{\cosh(-ah+c_1)}{a} + c_0 = 0$$

thus for c_1, c_2

$$-ac_0 = \cosh(ah+c_1) = \cosh(-ah+c_1)$$

because $\cosh(\cdot)$ is a even function, and is a strictly increasing function for $x > 0$

$$ah + c_1 = -ah + c_1 \text{ or } (ah+c_1) + (-ah+c_1) = 2c_1 = 0$$

here $ah \neq 0$, so $c_1 = 0,\ c_0 = -\frac{\cosh(ah)}{a}$

$$y = \frac{\cosh(ax) - \cosh(ah)}{a}$$

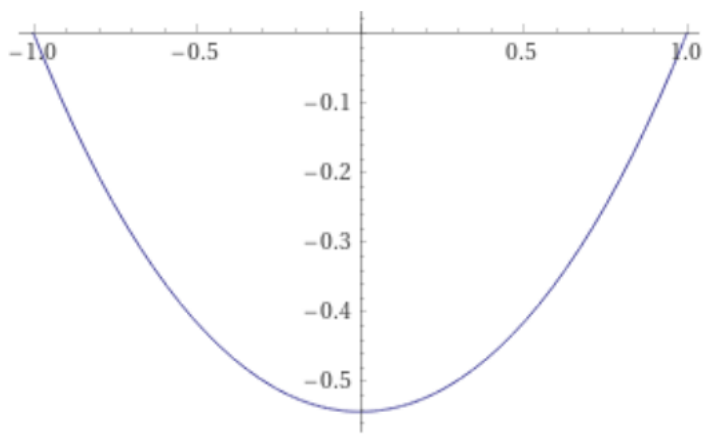

Figure 1. $y = \frac{\cosh(ax)-\cosh(ah)}{a}$, where $a = h = 1$

JOURNAL.

Compare and contrast the eigenvalue problems for square matrices and for differential operators. In particular, discuss the difference and similarity regarding the number of eigenvalues and the meaning of orthogonality.

solution
Differential operator $\mathcal{L}$
Think about the general equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y+\alpha(x)\frac{\mathrm{d}}{\mathrm{d}x}y+\beta(x)y=f(x)$$

We can all convert into this form

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right]+q(x)y=\lambda w(x)f(x)$$

We want to expand $y,\ f$ in such form

$$y=\sum_{n=0}^{\infty}c_ny_n,\ \ f(x)=\sum_{n=0}^{\infty}d_ny_n$$

To calculate c_n, assume: for eigenvalue $\lambda_n=d_n/c_n$, we have eigenfunction y_n
For the SL problem, here $x\in[a,b]$, the operator $\mathcal{L}$

$$\mathcal{L}y_n=-\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right]+q(x)y_n=\lambda_nw(x)y_n$$

$$\mathcal{L}y=\sum_{n=0}^{\infty}c_n\lambda_nw(x)y_n=w(x)\sum_{n=0}^{\infty}d_ny_n=w(x)f(x)$$

Now, if λ_n, y_n are known, we want to find the expression of d_n, then we can conclude $c_n=d_n/\lambda_n$
Guess there is a bracket operator $\langle\cdot,\cdot\rangle$

$$\langle f,y_n\rangle=d_n\langle y_n,y_n\rangle+\sum_{m\neq n}d_m\langle y_m,y_n\rangle$$

If $\langle y_m,y_n\rangle=0$ for $m\neq n$, we can conclude $d_n=\langle f,y_n\rangle/\langle y_n,y_n\rangle$
So, what the bracket operator $\langle\cdot,\cdot\rangle$ should be, let's guess, for $\lambda_n\neq\lambda_m$

$$\langle y_n,y_m\rangle=0\Leftrightarrow(\lambda_n-\lambda_m)\langle y_n,y_m\rangle=0\Leftrightarrow\langle\lambda_ny_n,y_m\rangle=\langle y_n,\lambda_my_m\rangle$$

think about the relationship $\lambda_nw(x)y_n=\mathcal{L}y_n$ notice that formula

$$\int_a^b y_n\frac{d}{dx}\left[p(x)\frac{dy_m}{dx}\right]dx=\left[y_np(x)\frac{dy_m}{dx}\right]_a^b-\int_a^b p(x)\frac{dy_n}{dx}\frac{dy_m}{dx}dx\ \textbf{symmetric form}$$

$$\int_a^b y_nq(x)y_mdx\ \textbf{symmetric form}$$

here

$$\int_a^b y_n\mathcal{L}y_m-y_m\mathcal{L}y_n=\left[y_mp(x)\frac{dy_n}{dx}-y_np(x)\frac{dy_m}{dx}\right]_a^b=\left[p(x)\left(y_m\frac{dy_n}{dx}-y_n\frac{dy_m}{dx}\right)\right]_a^b$$

If we have (i) $p(a) = p(b) = 0$, (ii) $p(a) = p(b), y_n(a) = y_n(b), y'_n(a) = y'_n(b)$,
(iii) $\alpha_1 y_n(a) + \alpha_2 y'_n(a) = 0, \beta_1 y_n(b) + \beta_2 y'_n(b) = 0$ for all n

$$\int_a^b y_n \mathcal{L} y_m - y_m \mathcal{L} y_n = \int_a^b y_n \lambda_m w(x) y_m - y_m \lambda_n w(x) y_n = 0$$

now we can define the bracket as

$$\langle u, v\rangle = \int_a^b u\ w(x)\ v dx, \quad \langle y_n, \lambda_m y_m\rangle = \int_a^b y_n w(x) \lambda_m y_m dx, \quad \langle \lambda_n y_n, y_m\rangle = \int_a^b \lambda_n y_n w(x) y_m dx$$

$$\langle \lambda_n y_n, y_m\rangle = \langle y_n, \lambda_m y_m\rangle$$

If λ_n has multiple eigenfunctions y_m, y'_n, **Gram–Schmidt process** can make sure $\langle y_n, y'_n\rangle = 0$

Square matrice A
For square matrix A

$$AX = b$$

we can first expand $X = \sum c_n X_n, b = \sum d_n X_n$, then assume we have independent equations

$$c_n A X_n = d_n X_n \Leftrightarrow A X_n = (d_n / c_n) X_n = \lambda_n X_n$$

once we know d_n, λ_n, we can determine $c_n = d_n/\lambda_n$ directly
especially, when $A^T = A$, for $\lambda_n \neq \lambda_m$

$$\lambda_n X_m^T X_n = X_m^T A X_n = X_m^T A^T X_n = \lambda_m X_m^T X_n \Leftrightarrow X_m^T X_n = 0$$

here we obtain d_n with

$$X_n^T b = X_n^T \sum d_n X_n = d_n X_n^T X_n \Rightarrow d_n = X_n^T b / X_n^T X_n$$

If λ_n has multiple eigenvectors X_n, X'_n, **Gram–Schmidt process** can make sure $X_n^T X'_n = 0$

Similarity
The operator $\mathcal{L}$, square matrix A, they both use the orthogonality:$\langle y_m, y_n\rangle = 0, X_m^T X_n = 0\ (m \neq n)$
to determine the coefficients c_n of components: eigenfunction y_n, eigenvector X_n

Difference
For the operator $\mathcal{L}$, it **could** have **infinite countable** eigenvalues
especially, when it satisfy (iii) $\alpha_1 y_n(a) + \alpha_2 y'_n(a) = 0, \beta_1 y_n(b) + \beta_2 y'_n(b) = 0$ for all n,
each λ_n only has one eigenfunction y_n,
otherwise, with eigenfunctions y_n, y'_n, (iii)$\Rightarrow W(a) = 0 \Rightarrow W(x) = 0 \Rightarrow y_n, y'_n$ linearly dependent
(note: λ_n could have multiple eigenfunctions f_n (e.g. section 5.2.2 Problem 3)
when it satisfies (i) $p(a) = p(b) = 0$ or (ii) $p(a) = p(b), y_n(a) = y_n(b), y'_n(a) = y'_n(b)$ instead of (iii))

For the square matrix $A = A^T$, it only has **finite** eigenvalues,
each λ_n could have multiple eigenfunction X_n