Problem 12

12. In Section 1.3 we obtained the initial value problem

$$\frac{d^2h}{dt^2} = -\frac{1}{(1+\varepsilon h)^2}, \quad h(0)=0, \quad h'(0)=1, \quad 0<\varepsilon \ll 1$$

governing the motion of a projectile. Use regular perturbation theory to obtain a three-term perturbation approximation. Up to the accuracy of ε^2 terms, determine the value t_m when h is maximum. Find $h_{\max} \equiv h(t_m)$ up through order ε^2 terms.
(Problem 12 on Page 167, PDF Page 199)

solution
Notice $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$, substitute $x = -\varepsilon h$

$$\frac{d^2h}{dt^2} = \sum_{n=0}^{\infty}(n+1)(-1)^{n+1}h^n\varepsilon^n$$

Expand $h(\varepsilon, t)$ with ε

$$h(\varepsilon,t) = y_0(t) + y_1(t)\varepsilon + y_2(t)\varepsilon^2 + \cdots$$

Substitute $h(\varepsilon,t) = y_0(t) + y_1(t)\varepsilon + y_2(t)\varepsilon^2 + \cdots$ in $\frac{d^2h}{dt^2} = \sum_{n=0}^{\infty}(n+1)(-1)^{n+1}h^n\varepsilon^n$
Compare the coefficients of $1, \varepsilon, \varepsilon^2$ respectively

$$\begin{aligned}
1 :& y_0'' = -1\\
\varepsilon :& y_1'' = (1+1)(-1)^{1+1}\binom{1}{1}y_0 = 2y_0\\
\varepsilon^2 :& y_2'' = (1+1)(-1)^{1+1}\binom{1}{1}y_1 + (2+1)(-1)^{2+1}\binom{2}{2}y_0^2 = 2y_1 - 3y_0^2
\end{aligned}$$

Initial condition $h(\varepsilon,0) = y_0(0) + y_1(0)\varepsilon + y_2(0)\varepsilon^2 + \cdots = 0$, compare the coefficients of $\varepsilon^k, k \in \mathrm{N}$

$$y_0(0) = y_1(0) = y_2(0) = \cdots = 0$$

Initial condition $h'(\varepsilon,0) = y_0'(0) + y_1'(0)\varepsilon + y_2'(0)\varepsilon^2 + \cdots = 1$, compare the coefficients of $\varepsilon^k, k \in \mathrm{N}$

$$y_0(0) = 1, \quad y_1(0) = y_2(0) = \cdots = 0$$

For y_0, solve $y_0'' = -1$

$$y_{0p} = -\frac{1}{2}t^2, y_{0h} = c_1 t + c_2, y_0 = y_{0p} + y_{0h} = -\frac{1}{2}t^2 + c_1 t + c_2$$

With initial conditions

$$y_0(0) = c_2 = 0, y_0'(0) = c_1 = 1 \Rightarrow c_1 = 1, c_2 = 0$$

$$y_0 = -\frac{1}{2}t^2 + t$$

For y_1, solve $y_1'' = 2y_0 = -t^2 + 2t$

$$y_{1p} = -\frac{1}{12}t^4 + \frac{1}{3}t^3, y_{1h} = c_1 t + c_2, y_0 = y_{1p} + y_{1h} = -\frac{1}{12}t^4 + \frac{1}{3}t^3 + c_1 t + c_2$$

With initial conditions

$$y_1(0) = c_2 = 0, y_1'(0) = c_1 = 0 \Rightarrow c_1 = 0, c_2 = 0$$

$$y_1 = -\frac{1}{12}t^4 + \frac{1}{3}t^3$$

For y_2, solve $y_2'' = 2y_1 - 3y_0^2 = -\frac{1}{6}t^4 + \frac{2}{3}t^3 - 3(-\frac{1}{2}t + 1)^2 t^2 = -\frac{11}{12}t^4 + \frac{11}{3}t^3 - 3t^2$

$$y_{2p} = -\frac{11}{360}t^6 + \frac{11}{60}t^5 - \frac{1}{4}t^4, y_{2h} = c_1 t + c_2, y_0 = y_{2p} + y_{2h} = -\frac{1}{12}t^4 + \frac{1}{3}t^3 + c_1 t + c_2$$

With initial conditions

$$y_2(0) = c_2 = 0, y_2'(0) = c_1 = 0 \Rightarrow c_1 = 0, c_2 = 0$$

$$y_2 = -\frac{11}{360}t^6 + \frac{11}{60}t^5 - \frac{1}{4}t^4$$

In all

$$\begin{aligned} y_0 &= -\frac{1}{2}t^2 + t \\ y_1 &= -\frac{1}{12}t^4 + \frac{1}{3}t^3 \\ y_2 &= -\frac{11}{360}t^6 + \frac{11}{60}t^5 - \frac{1}{4}t^4 \\ h(\varepsilon, t) &\approx y_0(t) + y_1(t)\varepsilon + y_2(t)\varepsilon^2 \\ &= \left[-\frac{1}{2}t^2 + t\right] + \varepsilon\left[-\frac{1}{12}t^4 + \frac{1}{3}t^3\right] + \varepsilon^2\left[-\frac{11}{360}t^6 + \frac{11}{60}t^5 - \frac{1}{4}t^4\right] \end{aligned} \tag{1}$$

Calculate the partial derivatives of function $h(\varepsilon, t)$

$$\begin{aligned} h(\varepsilon, t) &= \left[-\frac{1}{2}t^2 + t\right] + \varepsilon\left[-\frac{1}{12}t^4 + \frac{1}{3}t^3\right] + \varepsilon^2\left[-\frac{11}{360}t^6 + \frac{11}{60}t^5 - \frac{1}{4}t^4\right] + \cdots \\ \frac{\partial h(\varepsilon, t)}{\partial t} &= [-t + 1] + \varepsilon\left[-\frac{1}{3}t^3 + t^2\right] + \varepsilon^2\left[-\frac{11}{60}t^5 + \frac{11}{12}t^4 - t^3\right] + \cdots \\ \frac{\partial^2 h(\varepsilon, t)}{\partial t^2} &= [-1] + \varepsilon\left[-t^2 + 2t\right] + \varepsilon^2\left[-\frac{11}{12}t^4 + \frac{11}{3}t^3 - 3t^2\right] + \cdots \end{aligned}$$

For $t_{\max}(\varepsilon)$ should have (i) $\frac{\partial^2 h(\varepsilon,t)}{\partial t^2}|_{t=t_{\max}} < 0$ holds for $0 < \varepsilon \ll 1$; (ii) $\frac{\partial h(\varepsilon,t)}{\partial t}|_{t=t_{\max}} = 0$

$$t_{\max}(\varepsilon) = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$$

Substitute $t_{\max}(\varepsilon) = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$ into $\frac{\partial h(\varepsilon,t)}{\partial t}|_{t=t_{\max}} = 0$, compare coefficients of ε^k

$$\begin{aligned} &1: -a_0 + 1 = 0 \\ &\varepsilon: \left[-\binom{1}{1}a_1\right] + \left[-\frac{1}{3}\binom{3}{3}a_0^3 + \binom{2}{2}a_0^2\right] = 0 \\ &\varepsilon^2: \left[-\binom{1}{1}a_2\right] + \left[-\frac{1}{3}\binom{3}{2}\binom{1}{1}a_0^2 a_1 + \binom{2}{1}\binom{1}{1}a_0 a_1\right] + \left[-\frac{11}{60}\binom{5}{5}a_0^5 + \frac{11}{12}\binom{4}{4}a_0^4 - \binom{3}{3}a_0^3\right] = 0 \end{aligned}$$

For 1, we have $a_0 = 1$

For ε, we have

$$-a_1 + \frac{2}{3} = 0 \Rightarrow a_1 = \frac{2}{3}$$

For ε^2, we have

$$-a_2 + \left[-\frac{2}{3} + \frac{4}{3}\right] + \left[-\frac{11}{60} + \frac{11}{12} - 1\right] = 0 \Rightarrow a_2 = \frac{2}{5}$$

To sum up, for $t_{\max}(\varepsilon)$

$$t_{\max}(\varepsilon) = 1 + \frac{2}{3}\varepsilon + \frac{2}{5}\varepsilon^2 + \cdots$$

Substitute $t_{\max}(\varepsilon) = 1 + \frac{2}{3}\varepsilon + \frac{2}{5}\varepsilon^2 + \cdots$ into $h(\varepsilon, t)|_{t=t_{\max}}$, calculate coefficients of ε^k

$$1 : -\frac{1}{2}\binom{2}{2}1^2 + 1 = \frac{1}{2}$$

$$\varepsilon : \left[-\frac{1}{2}\binom{2}{1}\binom{1}{1}1 \cdot \frac{2}{3} + \binom{1}{1}\frac{2}{3}\right] + \left[-\frac{1}{12}\binom{4}{4}1^4 + \frac{1}{3}\binom{3}{3}1^3\right] = \frac{1}{4}$$

$$\begin{aligned}\varepsilon^2 : & \left[-\frac{1}{2}\left[\binom{2}{1}\binom{1}{1}1 \cdot \frac{2}{5} + \binom{2}{2}\left(\frac{2}{3}\right)^2\right] + \binom{1}{1}\frac{2}{5}\right] + \left[-\frac{1}{12}\binom{4}{3}\binom{1}{1}1^3 \cdot \frac{2}{3} + \frac{1}{3}\binom{3}{2}\binom{1}{1}1^2 \cdot \frac{2}{3}\right] + \\ & \left[-\frac{11}{360}\binom{6}{6}1^6 + \frac{11}{60}\binom{5}{5}1^5 - \frac{1}{4}\binom{4}{4}1^4\right] \\ & = [-\frac{1}{2} \cdot \frac{56}{45} + \frac{2}{5}] + \frac{4}{9} - \frac{7}{72} = -\frac{2}{9} + \frac{4}{9} - \frac{7}{72} = \frac{1}{8}\end{aligned}$$

Finally, for the $t_m = t_{\max}(\varepsilon)$, $h_{\max} = h(\varepsilon, t)|_{t=t_{\max}}$

$$t_m = 1 + \frac{2}{3}\varepsilon + \frac{2}{5}\varepsilon^2 + \cdots, \quad h_{\max} = \frac{1}{2} + \frac{1}{4}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots \tag{2}$$

We can verify it with MATLAB code, for example when $\varepsilon = 0.02$

```
clear; clc; close all
eps = 0.02;
dgdt = @(t, h) -1./ (1 + eps*h).^2; % g'= h; h''=g'=-1./ (1+eps*h).^2
func = @(t, sol) [sol(2,:); dgdt(t, sol(1,:))];
h0 = 0; g0 = 1; sol0 = [h0 g0];
tspan = (0:0.000002:1.2)';
[t,sol] = ode23(func, tspan, sol0);
[h, g] = deal(sol(:, 1), sol(:, 2));
plot(t,h,'b-'); xlabel('$t$','Interpreter','latex');
ylabel('$h$','Interpreter','latex'); grid on;
title('$h(\varepsilon, t)$','Interpreter','latex');
[h_max, ind] = max(h);              % find h_max
t_m = t(ind);                       % find t_m
fprintf('numerical   :t_m=%.6f, h_max=%.6f\n', t_m, h_max);
func_tm = @(eps) 1 + (2/3) * eps + (2/5) * eps^2;
func_hmax = @(eps) 1/2 + (1/4) * eps + (1/8) * eps^2;
fprintf('theoretical:t_m=%.6f, h_max=%.6f\n', ...
func_tm(eps),func_hmax(eps));
```

The numerical results and theoretical results when $\varepsilon = 0.02$

```
numerical   :t_m=1.013498, h_max=0.505053
theoretical:t_m=1.013493, h_max=0.505050
```

Problem 1

1. A homogeneous bar of length 40 cm has its left and right ends held at 30 °C and 10 °C, respectively. If the temperature in the bar is in steady state,what is the temperature in the cross-section 12 cm from the left end? If the thermal conductivity is K, what is the rate that heat is leaving the bar at its right face?
(Problems 1 on Page 283, PDF Page 332)

solution
Write the physic law

$$-\left(K(x)u'(x)\right)' = f(x), \quad 0 < x < 40$$
$$u(0) = 30,\ u(40) = 10$$

heat flux $\phi(x) \equiv -K(x)u'(x)$ at x (the rate that heat is leaving in the right direction at x)
here $K(x) = K,\ f(x) = 0$

$$-Ku''(x) = 0, \quad 0 < x < 40$$
$$u(0) = 30,\ u(40) = 10$$

Set $u(x) = c_0 + c_1 x$, consider the boundary conditions

$$u(0) = c_0 = 30,\ u(40) = c_0 + c_1 \cdot 40 = 10 \Rightarrow c_0 = 30,\ c_1 = -\frac{1}{2}$$

Thus

$$u(x) = 30 - \frac{1}{2}x, \quad 0 \le x \le 40$$

(1) the temperature in the cross-section 12 cm from the left end

$$u(12) = 30 - \frac{1}{2} \cdot 12 = 24 \tag{1}$$

(2) the rate that heat is leaving the bar at its right face
that is heat flux $\phi(x) \equiv -K(x)u'(x) = -Ku'(x)$ at $x = 40$

$$\phi(x)|_{x=40} = -Ku'(x)|_{x=40} = -K\left(-\frac{1}{2}\right) = \frac{K}{2} \tag{2}$$

PROBLEM 2

2. The thermal conductivity of a bar of length $L = 20$ and cross-sectional area $A = 2$ is $K(x) = 1$, and an internal heat source is given by $f(x) = 0.5x(L - x)$. If both ends of the bar are maintained at zero degrees, what is the steady-state temperature distribution in the bar? Sketch a graph of $u(x)$. What is the rate that heat is leaving the bar at $x = 20$?
(Problems 2 on Page 283, PDF Page 332)

solution
Write the physic law

$$-\left(K(x)u'(x)\right)' = f(x), \quad 0 < x < 20$$
$$u(0) = 0,\ u(20) = 0$$

heat flux $\phi(x) \equiv -K(x)u'(x)$ at x (the rate that heat is leaving in the right direction at x)
here $K(x) = 1,\ f(x) = 0.5x(20 - x)$

$$-u''(x) = \frac{1}{2}x(20 - x), \quad 0 < x < 20$$
$$u(0) = 0,\ u(20) = 0$$

Integrate for 2 times

$$u = \int \left(\int u'' dx \right) dx = \int \left(\int -\frac{1}{2}x(20 - x)dx \right) dx = \int \left[\frac{1}{6}x^3 - 5x^2 + c_1 \right] dx = \frac{1}{24}x^4 - \frac{5}{3}x^3 + c_1 x + c_0$$

Consider the boundary conditions

$$u(0) = c_0 = 0,\ u(20) = \frac{1}{24}20^4 - \frac{5}{3}20^3 + c_1 20 + c_0 = 0 \Rightarrow c_1 = \frac{2000}{6},\ c_0 = 0$$

(1) the steady-state temperature distribution in the bar

$$u = \frac{1}{24}x^4 - \frac{5}{3}x^3 + \frac{2000}{6}x, \quad 0 \le x \le 20 \tag{1}$$

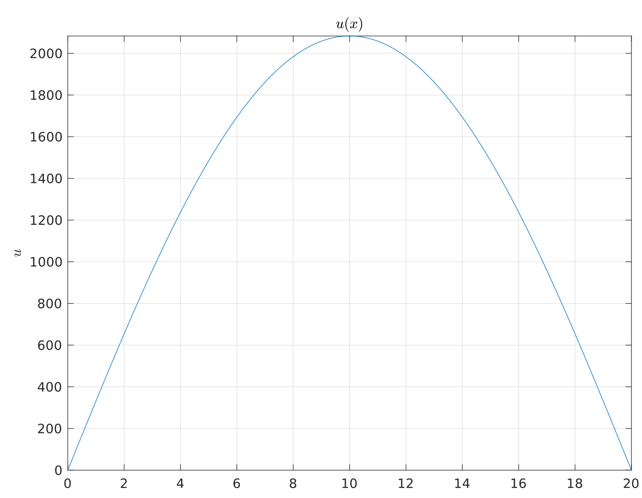

FIGURE 1. the graph of $u(x)$

```
clear; clc; close all
func = @(x) (1/24) * x.^4 - (5/3) * x.^3 + (2000/6) * x;
x = (0:0.01:20)';
u = func(x);
plot(x, u); ylim([0, inf]); grid on;
xlabel('$x$','Interpreter','latex');
ylabel('$u$','Interpreter','latex');
title('$u(x)$','Interpreter','latex');
```

(2) the rate that heat is leaving the bar at $x = 20$
that is heat flux $\phi(x) \equiv -K(x)u'(x) = -u'(x)$ at $x = 20$

$$\phi(x) = -u'(x) = -\left[\frac{1}{6}x^3 - 5x^2 + \frac{2000}{6}\right]$$

Thus

$$\phi(x)|_{x=20} = -\left[\frac{1}{6}20^3 - 5 \cdot 20^2 + \frac{2000}{6}\right] = -\frac{2000}{6} \tag{2}$$

Problem 5

5. Consider the nonlinear heat flow problem

$$(uu')' = 0, \quad 0 < x < \pi$$
$$u(0) = 0, \quad u'(\pi) = 1$$

where the thermal conductivity depends on temperature and is given by $K(u) = u$. Find the steady-state temperature distribution.
(Problems 5 on Page 283, PDF Page 332)

solution
Firstly set the coefficient c_0

$$(uu')' = 0 \Leftrightarrow uu' = c_0 \Leftrightarrow udu = c_0 dx$$

Integrate on both sides

$$\frac{1}{2}u^2 = \int udu = \int c_0 dx = c_0 x + c_1$$

For $u(0) = 0$

$$\frac{1}{2}0^2 = c_0 \cdot 0 + c_1 \tag{1}$$

Because of $uu' = c_0$

$$(uu')^2 = c_0^2$$
$$2\left(u'\right)^2 = \frac{(uu')^2}{\frac{1}{2}u^2} = \frac{c_0^2}{c_0 x + c_1}$$

For $u'(\pi) = 1$

$$2 \cdot 1^2 = \frac{c_0^2}{c_0 \pi + c_1} \tag{2}$$

Combine (1), (2)

$$c_0 = 2\pi, c_1 = 0$$

Thus is

$$\frac{1}{2}u^2 = 2\pi x \Leftrightarrow u^2 = 4\pi x \Rightarrow u = +\sqrt{4\pi x}, \quad 0 \le x \le \pi \tag{3}$$

Bonus.

1. In a spring-mass problem assume that the restoring force is $-ky$ and that there is a resistive force numerically equal to $a\dot{y}^2$, where k and a are constants with appropriate units. With initial conditions $y(0) = A, \dot{y}(0) = 0$, determine the correct time and displacement scales for small damping and show that the problem can be written in dimensionless form as

$$\begin{gathered} \bar{y}'' + \varepsilon\left(\bar{y}'\right)^2 + \bar{y} = 0 \\ \bar{y}(0) = 1, \quad \bar{y}'(0) = 0 \end{gathered}$$

where $\varepsilon \equiv aA/m$ is a dimensionless parameter and prime denotes the derivative with respect to the scaled time $\bar{t}$.
In addition, find a two-term approximation for $0 < \varepsilon \ll 1$. (Note: $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$)
(Problem 1 on Page 165, PDF Page 197)

solution
We can write the physical law as

$$\begin{gathered} m\frac{d^2y}{dt^2} + a\left(\frac{dy}{dt}\right)^2 + ky = 0 \\ y(0) = A, \quad \frac{dy}{dt}|_{t=0} = 0 \end{gathered}$$

Now we define $\bar{y} = y/y_c, \bar{t} = t/t_c$, it becomes

$$\begin{gathered} \left[\frac{y_c}{t_c^2}\right] m\frac{d^2\bar{y}}{d\bar{t}^2} + \left[\frac{y_c^2}{t_c^2}\right] a\left(\frac{d\bar{y}}{d\bar{t}}\right)^2 + [y_c]\, k\bar{y} = 0 \\ [y_c]\, \bar{y}(0) = A, \quad \frac{d\bar{y}}{d\bar{t}}|_{\bar{t}=0} = 0 \end{gathered}$$

That is

$$\begin{gathered} \frac{d^2\bar{y}}{d\bar{t}^2} + [y_c]\, \frac{a}{m}\left(\frac{d\bar{y}}{d\bar{t}}\right)^2 + [t_c^2]\, \frac{k}{m}\bar{y} = 0 \\ \bar{y}(0) = \left[\frac{1}{y_c}\right] A, \quad \frac{d\bar{y}}{d\bar{t}}|_{\bar{t}=0} = 0 \end{gathered}$$

Compare the coefficients of $\bar{y}, \bar{y}(0)$ we have

$$[t_c^2]\, \frac{k}{m} = 1, \quad \left[\frac{1}{y_c}\right] A = 1$$

So, the characteristic scale y_c, t_c

$$t_c = \sqrt{\frac{m}{k}}, \quad y_c = A \tag{1}$$

Then equation becomes

$$\begin{gathered} \frac{d^2\bar{y}}{d\bar{t}^2} + \frac{aA}{m}\left(\frac{d\bar{y}}{d\bar{t}}\right)^2 + \bar{y} = 0 \\ \bar{y}(0) = 1, \quad \frac{d\bar{y}}{d\bar{t}}|_{\bar{t}=0} = 0 \end{gathered}$$

here $\varepsilon \equiv \frac{aA}{m}$, that is

$$\begin{gathered} \frac{d^2\bar{y}}{d\bar{t}^2} + \varepsilon\left(\frac{d\bar{y}}{d\bar{t}}\right)^2 + \bar{y} = 0 \\ \bar{y}(0) = 1, \quad \frac{d\bar{y}}{d\bar{t}}|_{\bar{t}=0} = 0 \end{gathered}$$

Expand the $\bar{y}(\varepsilon, \bar{t})$

$$\bar{y}(\varepsilon, \bar{t}) = y_0(\bar{t}) + y_1(\bar{t})\varepsilon + \cdots$$

Substitute $\bar{y}(\varepsilon, \bar{t}) = y_0(\bar{t}) + y_1(\bar{t})\varepsilon + \cdots$ into equation, compare the coefficients of ε^k

$$1 : y_0'' + y_0 = 0$$

$$\varepsilon : y_1'' + (y_0')^2 + y_1 = 0$$

For the initial conditions

$$\bar{y}(0) = y_0(0) + y_1(0)\varepsilon + \cdots = 1, \quad \bar{y}'(0) = y_0'(0) + y_1'(0)\varepsilon + \cdots = 0$$

Compare the coefficients of ε^k

$$y_0(0) = 1, y_1(0) = \cdots = 0$$
$$y_0'(0) = y_1'(0) = \cdots = 0$$

For $y_0(\bar{t})$, solve

$$y_0'' + y_0 = 0, \quad y_0(0) = 1, y_0'(0) = 0$$

Set $y_0(\bar{t}) = c_1 \sin(\bar{t}) + c_2 \cos(\bar{t})$

$$y_0(0) = c_2 = 1, y_0'(0) = c_1 = 0 \Rightarrow c_1 = 0, c_2 = 1$$
$$y_0(\bar{t}) = \cos(\bar{t})$$

For $y_1(\bar{t})$, solve

$$y_1'' + y_1 = -(y_0')^2 = -\frac{1}{2} + \frac{1}{2}\cos(2\bar{t}), \quad y_1(0) = 0, y_1'(0) = 0$$

Set $y_1(\bar{t}) = y_{1p}(\bar{t}) + c_1 \sin(\bar{t}) + c_2 \cos(\bar{t})$, for $y_{1p}(\bar{t}) = -\frac{1}{2} + c_0 \cos(2\bar{t})$

$$(-4c_0 + c_0)\cos(2\bar{t}) = \frac{1}{2}\cos(2\bar{t}) \Rightarrow c_0 = -\frac{1}{6}$$

so, $y_1(\bar{t}) = -\frac{1}{2} - \frac{1}{6}\cos(2\bar{t}) + c_1 \sin(\bar{t}) + c_2 \cos(\bar{t})$

$$y_1(0) = -\frac{1}{2} - \frac{1}{6} + c_2 = 0, y_1'(0) = c_1 = 0 \Rightarrow c_1 = 0, c_2 = \frac{2}{3}$$
$$y_1(\bar{t}) = -\frac{1}{2} - \frac{1}{6}\cos(2\bar{t}) + \frac{2}{3}\cos(\bar{t})$$

To sum up

$$\begin{aligned} y_0(\bar{t}) &= \cos(\bar{t}) \\ y_1(\bar{t}) &= -\frac{1}{2} + \frac{2}{3}\cos(\bar{t}) - \frac{1}{6}\cos(2\bar{t}) \\ \bar{y}(\varepsilon, \bar{t}) &= y_0(\bar{t}) + y_1(\bar{t})\varepsilon + \cdots \\ &= \cos(\bar{t}) + \varepsilon\left[-\frac{1}{2} + \frac{2}{3}\cos(\bar{t}) - \frac{1}{6}\cos(2\bar{t})\right] + \cdots \end{aligned} \tag{2}$$

Journal.

Compare and contrast initial value problems and boundary value problems. In particular, explain how many initial/boundary conditions are needed, and what are different types of initial/boundary conditions.

solution
If we have N **undetermined coefficients** in equations, then N initial/boundary conditions are needed.

For the equation

$$(p(x)y)' + q(x)y = \lambda y, \quad a < x < b$$

The equation is usually accompanied by homogeneous boundary conditions on y(x) of the form

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0$$

Especially, two special cases of the boundary conditions are

$$y(a) = 0, y(b) = 0, \quad \text{(Dirichlet conditions]}$$
$$y'(a) = 0, y'(b) = 0. \quad \text{(Ncumann conditions)}$$