Problem 4

4. Find the values of μ where solutions bifurcate and examine the stability of the origin in each case. (Problems 4af on Page 108, PDF Page 131)

a) $x' = x + \mu y, y' = \mu x + y$
f) $x' = y, y' = x^2 - x + \mu y$

solution
a) The nullclines: $0 = x + \mu y, 0 = \mu x + y$
Of course, $(0, 0)$ is one critical point

$$A = \begin{pmatrix} 1 & \mu \\ \mu & 1 \end{pmatrix}, \quad |A - \lambda I| = \lambda^2 - p\lambda + q = \lambda^2 - 2\lambda + (1 - \mu^2) = 0$$

With **Vieta theorem**, $\Delta = 2^2 - 4 \times (1 - \mu^2) = 4\mu^2, \lambda_1 + \lambda_2 = p = 2 > 0, \lambda_1\lambda_2 = q = 1 - \mu^2$

For different values of μ
condition 1: $q = 0, \Delta \neq 0$, one eigenvalue $\lambda_1 = 0$
$q = 1 - \mu^2, \Delta = 4\mu^2 \neq 0 \Leftrightarrow \mu = \pm 1$
the other eigenvalue $\lambda_2 = p = 2 > 0$
critical point is a **unstable borderline**

condition 2: $\Delta = 0, q \neq 0$, repeated real eigenvalues $\lambda_1 = \lambda_2 = \frac{p}{2}$
$\Delta = 4\mu^2 = 0, q = 1 - \mu^2 \neq 0 \Leftrightarrow \mu = 0$
repeated real eigenvalues $\lambda_1 = \lambda_2 = \frac{p}{2} = 1 > 0$
critical point is a **unstable node**
$(A - \lambda I)w = 0$ eigenvector w_1, w_2: **proper node: star** $X = (c_1 w_1 + c_2 w_2)e^{\lambda t}$

condition 3: $\Delta > 0, q > 0$, eigenvalues are both positive $\lambda_1 > 0, \lambda_2 > 0$
$\Delta = 4\mu^2 > 0, q = 1 - \mu^2 > 0 \Leftrightarrow -1 < \mu < 1, \mu \neq 0$
critical point is an **unstable node**

condition 4: $\Delta > 0, q < 0$, signs fo eigenvalues are opposite $\lambda_1 > 0, \lambda_2 < 0$
$\Delta = 4\mu^2 > 0, q = 1 - \mu^2 < 0 \Leftrightarrow \mu < -1$ or $1 < \mu$
critical point is a **unstable saddle**

f) The nullclines: $0 = y, 0 = x^2 - x + \mu y$, of course, $(0,0)$ is one critical point
Consider the Jacobian J, at the critical point $(0,0)$

$$J = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2x-1 & \mu \end{pmatrix}, \quad J_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

$$|J - \lambda I| = \lambda^2 - p\lambda + q = \lambda^2 - \mu\lambda + 1 = 0$$

With **Vieta theorem**, $\Delta = \mu^2 - 4 \times 1 = \mu^2 - 4, \lambda_1 + \lambda_2 = p = \mu > 0, \lambda_1\lambda_2 = q = 1$
condition 1: $\Delta = 0, q \neq 0$, repeated real eigenvalues $\lambda_1 = \lambda_2 = \frac{p}{2}$
$\Delta = \mu^2 - 4 = 0, q = 1 \neq 0 \Leftrightarrow \mu = \pm 2$

- $\mu = -2$ repeated real eigenvalues $\lambda_1 = \lambda_2 = \frac{p}{2} = \frac{\mu}{2} = -1 < 0$
  critical point is a **stable node**
  $(J - \lambda I)w = 0, (J - \lambda I)v = w$: **improper node** $X = c_1 w e^{\lambda t} + c_2(w + vt)e^{\lambda t}$
- $\mu = +2$ repeated real eigenvalues $\lambda_1 = \lambda_2 = \frac{p}{2} = \frac{\mu}{2} = +1 > 0$
  critical point is an **unstable node**
  $(J - \lambda I)w = 0, (J - \lambda I)v = w$: **improper node** $X = c_1 w e^{\lambda t} + c_2(w + vt)e^{\lambda t}$

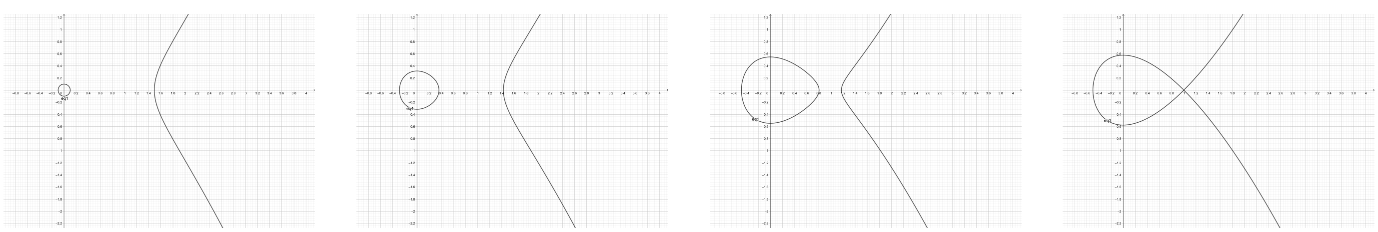
condition 2: $\Delta > 0, q > 0$, eigenvalues have the same signs $\lambda_1 < \lambda_2 < 0$ or $\lambda_1 > \lambda_2 > 0$
$\Delta = \mu^2 - 4 > 0, q = 1 > 0 \Leftrightarrow \mu < -2$ or $2 < \mu$

- $p = \mu < 0 \Rightarrow \mu < -2$ real eigenvalues $\lambda_1 < \lambda_2 < 0$
  critical point is a **stable node**
- $p = \mu > 0 \Rightarrow +2 < \mu$ real eigenvalues $\lambda_1 > \lambda_2 > 0$
  critical point is an **unstable node**

condition 3: $\Delta < 0, p = 0$, real part of eigenvalues $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \frac{p}{2} = 0$
$\Delta = \mu^2 - 4 < 0, p = \mu = 0 \Leftrightarrow \mu = 0$
we can't determine if the **spiral** is stable with Jacobian
solve $x' = y, y' = x^2 - x \Rightarrow x'' = x^2 - x$, with the substitution $x'' = \frac{dx'}{dx}\frac{dx}{dt} = \frac{dy}{dx}y$

$$\frac{dy}{dx}y = x^2 - x \Leftrightarrow \int y dy = \int (x^2 - x) dx \Leftrightarrow x^2 + y^2 - \frac{2}{3}x^3 = C$$

critical point is a **stable center**



(A) $C = 0.01$ (B) $C = 0.1$ (C) $C = 0.3$ (D) $C = \frac{1}{3}$

FIGURE 1. the diagram of $p, \frac{dp}{d\tau}$

condition 4: $\Delta < 0, p > 0$, real part of eigenvalues are both positive $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \frac{p}{2} > 0$
$\Delta = \mu^2 - 4 < 0, p = \mu > 0 \Leftrightarrow 0 < \mu < 2$
critical point is an **unstable spiral**
condition 5: $\Delta < 0, p > 0$, real part of eigenvalues are both negative $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \frac{p}{2} < 0$
$\Delta = \mu^2 - 4 < 0, p = \mu < 0 \Leftrightarrow -2 < \mu < 0$
critical point is a **stable spiral**

Problem 10

10. (Ecology) Let P denote the carbon biomass of plants in an ecosystem, H be the carbon biomass of herbivores, and ϕ the rate of primary carbon production in plants due to photosynthesis. A model of plant-herbivore dynamics is given by (Problem 10 on Page 109, PDF Page 133)

$$P' = \phi - aP - bHP$$
$$H' = \varepsilon bHP - cH$$

where a, b, c, and ε are positive parameters.

a) Explain the various terms and parameters in the model and determine the dimensions of each parameter.
b) Non-dimensionalize the model and find the equilibrium solutions.
c) Analyze the dynamics in the cases of high primary production ($\phi > ac/\varepsilon b$) and low primary production ($\phi < ac/\varepsilon b$). Explain what happens in the system if primary production is slowly increased from a low value to a high value.

solution

a) For the various terms, $[P] = [H] = M, [\phi] = MT^{-1}$
For the parameters a, b, c, $[aP] = [a]M = MT^{-1}, [bHP] = [b]M^2 = MT^{-1}, [cH] = [c]M = MT^{-1}$, thus $[a] = T^{-1}, [b] = M^{-1}T^{-1}, [c] = T^{-1}$
For the parameter ε, $[\varepsilon bHP] = [\varepsilon]M^{-1}T^{-1}M^2 = MT^{-1}$, thus $[\varepsilon] = 1$ is dimesionless

b) Non-dimensionalize the model
Firstly, the characteristic scale t_c to make the coefficient of cH becomes 1

$$\frac{1}{t_c} = c \Rightarrow t_c \equiv \frac{1}{c}$$

Secondly, the P_c to make the coefficient of εbHP becomes 1

$$\frac{1}{t_c} = \varepsilon b P_c \Rightarrow P_c \equiv \frac{c}{\varepsilon b}$$

Next, the H_c to make the coefficient of bHP becomes 1

$$\frac{1}{t_c} = bH_c \Rightarrow H_c \equiv \frac{c}{b}$$

In the end, with the substitution $\tau \equiv t/t_c, p = P/P_c, h \equiv H/H_c$

$$\frac{dp}{d\tau} = -\left(\frac{a}{c}\right)(p - \frac{\phi}{[ac/\varepsilon b]}) - hp$$
$$\frac{dh}{d\tau} = hp - h$$

Define parameters $\gamma \equiv \frac{a}{c}, \phi_{th} \equiv [ac/\varepsilon b], \varphi \equiv \frac{\phi}{\phi_{th}}$

$$\begin{aligned} \frac{dp}{d\tau} &= -\gamma(p - \frac{\phi}{\phi_{th}}) - hp = -\gamma(p - \varphi) - hp \\ \frac{dh}{d\tau} &= hp - h \end{aligned} \tag{1}$$

c) The nullclines of h: $0 = -\gamma(p-\varphi) - hp \Leftrightarrow h = \gamma(-1 + \frac{\varphi}{p})$
the nullclines of p: $0 = hp - h \Leftrightarrow h = 0, p = 1$
Consider the Jacobian J

$$J = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} -\gamma - h & -p \\ h & p-1 \end{pmatrix}$$

condition 1 $\varphi \equiv \frac{\phi}{\phi_{th}} \equiv \frac{\phi}{[ac/\varepsilon b]} = 1$
the critical point $(1, 0)$ for all $p \geq 0, h \geq 0$

$$J = \begin{pmatrix} -\gamma & -1 \\ 0 & 0 \end{pmatrix}, \quad \det(J) = 0, |J - \lambda I| = \lambda^2 + \gamma\lambda = 0$$

eigenvalues $\lambda_1 = 0, \lambda_2 = -\gamma < 0$ the equilibria $(1, 0)$ has the degenerated type:**concentrated in a line** $-\gamma(p-1) - h = 0$ (the line go through critical point), is a **stable borderline**

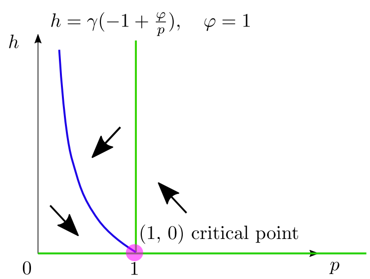


FIGURE 2. the phase diagram of $\varphi = 1$

condition 2 $0 < \varphi \equiv \frac{\phi}{\phi_{th}} \equiv \frac{\phi}{[ac/\varepsilon b]} < 1$
the critical point $(\varphi, 0)$ for all $p \geq 0, h \geq 0$

$$J = \begin{pmatrix} -\gamma & -\varphi \\ 0 & \varphi - 1 \end{pmatrix}, \quad |J - \lambda I| = [\lambda - (-\gamma)][\lambda - (\varphi - 1)] = 0$$

eigenvalues $\lambda_1 = \varphi - 1 < 0, \lambda_2 = -\gamma < 0$ the critical point $(\varphi, 0)$ is a **stable node**

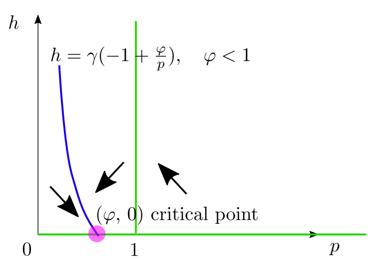


FIGURE 3. the phase diagram of $0 < \varphi < 1$

condition 3 $1 < \varphi \equiv \frac{\phi}{\phi_{th}} \equiv \frac{\phi}{[ac/\varepsilon b]}$
the critical points $(\varphi, 0), (1, \gamma(\varphi - 1))$ for all $p \geq 0, h \geq 0$

- At $(\varphi, 0)$

$$J = \begin{pmatrix} -\gamma & -\varphi \\ 0 & \varphi - 1 \end{pmatrix}, \quad |J - \lambda I| = [\lambda - (-\gamma)][\lambda - (\varphi - 1)] = 0$$

eigenvalues $\lambda_1 = \varphi - 1 > 0, \lambda_2 = -\gamma < 0$ the critical point $(\varphi, 0)$ is a **unstable saddle**

- At $(1, \gamma(\varphi - 1))$

$$J = \begin{pmatrix} -\gamma\varphi & -1 \\ \gamma(\varphi-1) & 0 \end{pmatrix}, \quad |J - \lambda I| = \lambda^2 + \gamma\varphi\lambda + \gamma(\varphi - 1) = \lambda^2 - p\lambda + q = 0$$

With **Vieta theorem** $p = \lambda_1 + \lambda_2 = -\gamma\varphi < 0, q = \lambda_1\lambda_2 = \gamma(\varphi - 1) > 0, \Delta = (\gamma\varphi)^2 - 4\gamma(\varphi - 1) = (\gamma\varphi - 2)^2 + 4(\gamma - 1)$

the critical point $(1, \gamma(\varphi - 1))$ is always **stable**

(1) $\gamma \equiv \frac{a}{c} = 1$

(a) if $\varphi = \frac{2}{\gamma} = 2$
the repeated eigenvalues $\lambda_1 = \lambda_2 = -1$, like $c_1 w e^{\lambda_1 t} + c_2(w + vt)e^{\lambda_1 t}$
the critical point $(1, \gamma(\varphi - 1)) = (1, 1)$ is a **stable improper node**

(b) if $\varphi \neq \frac{2}{\gamma} = 2$
hence, $\Delta > 0$ always holds, 2 real eigenvalues $\lambda_1 < \lambda_2 < 0$
the critical point $(1, \gamma(\varphi - 1))$ is a **stable node**

(2) $\gamma \equiv \frac{a}{c} > 1$
hence, $\Delta > 0$ always holds, it has 2 real eigenvalues $\lambda_1 < \lambda_2 < 0$
the critical point $(1, \gamma(\varphi - 1))$ is a **stable node**

(3) $0 < \gamma \equiv \frac{a}{c} < 1$

(a) if $\varphi = \frac{2(1 \pm \sqrt{1-\gamma})}{\gamma}$
hence $\Delta = 0$, the repeated eigenvalues $\lambda_1 = \lambda_2 = -1 \mp \sqrt{1 - \gamma}$, solution is in from of $c_1 w e^{\lambda_1 t} + c_2(w + vt)e^{\lambda_1 t}$
the critical point $(1, \gamma(\varphi - 1)) = (1, 2 - \gamma \pm 2\sqrt{1 - \gamma})$ is a **stable improper node**

(b) if $\frac{2(1-\sqrt{1-\gamma})}{\gamma} < \varphi < \frac{2(1+\sqrt{1-\gamma})}{\gamma}$
hence $\Delta < 0$, the complex eigenvalues $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \frac{p}{2} = -\gamma\varphi/2 < 0$
the critical point $(1, \gamma(\varphi - 1))$ is a **stable spiral**

(c) if $0 < \varphi < \frac{2(1-\sqrt{1-\gamma})}{\gamma}$ or $\frac{2(1+\sqrt{1-\gamma})}{\gamma} < \varphi$
hence $\Delta > 0$, 2 real eigenvalues $\lambda_1 < \lambda_2 < 0$
the critical point $(1, \gamma(\varphi - 1))$ is a **stable node**

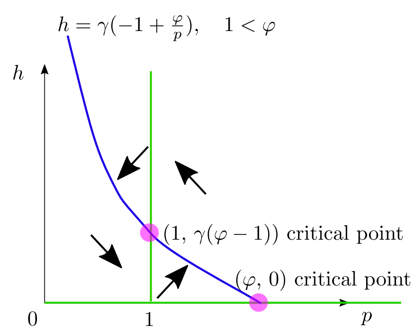


FIGURE 4. the phase diagram of $1 < \varphi$

As ϕ increases from low to high, the equilibrium of the biomass (P^*, H^*) changes as follows:

- For $0 < \varphi \equiv \frac{\phi}{[ac/\varepsilon b]} \leq 1$
stable critical point $(p, h) = (\varphi, 0) \Leftrightarrow$ equilibria $(P^*, H^*) = (\varphi P_c, 0) = (\frac{\phi}{[ac/\varepsilon b]}\frac{c}{\varepsilon b}, 0) = (\frac{\phi}{a}, 0)$
- For $1 < \varphi \equiv \frac{\phi}{[ac/\varepsilon b]}$
stable critical point $(p, h) = (1, \gamma(\varphi - 1))$
$\Leftrightarrow$ equilibria $(P^*, H^*) = (P_c, \gamma(\varphi - 1)H_c) = (\frac{c}{\varepsilon b}, \frac{a}{c}(\frac{\phi}{[ac/\varepsilon b]} - 1)\frac{c}{b}) = (\frac{c}{\varepsilon b}, \frac{\varepsilon\phi}{c} - \frac{a}{b})$

Problem 9

To find approximations to the roots of the cubic equation

$$x^3 - 4.001x + 0.002 = 0$$

why is it easier to examine the equation (Problem 9 on Page 166, PDF Page 199)

$$x^3 - (4+\varepsilon)x + 2\varepsilon = 0$$

- Find a two-term approximation to this equation.
- Then use software (e.g. matlab) to solve the above equation and compare the results.

solution

Perturbation is successful when $x_{approx.}(\varepsilon) - x(\varepsilon) \to 0$ at some well-defined rate as $\varepsilon \to 0$

We can find 3 solutions for $x^3-(4+\varepsilon)x+2\varepsilon = 0$ when $\varepsilon = 0$, that is $x_0 = 0, -2, +2$, corresponding to 3 continuous functions $x = f_1(\varepsilon), f_2(\varepsilon), f_3(\varepsilon)$, and $f_1(0) = 0, f_2(0) = -2, f_3(0) = +2$

Expand $x = f_i(\varepsilon), i = 1, 2, 3$

$$f_i(\varepsilon) = x_0 + x_1\varepsilon + x_2\varepsilon^2 + \cdots$$

Substitute $x = f_i(x) = x_0 + x_1\varepsilon + x_2\varepsilon^2 + \cdots$ in $x^3 - (4+\varepsilon)x + 2\varepsilon = 0$

Compare the coefficients of $1, \varepsilon, \varepsilon^2$ respectively

$$\begin{aligned} 1 :& x_0^3 - 4x_0 = 0 \\ \varepsilon :& \binom{3}{2} x_0^2 x_1 - 4x_1 - x_0 + 2 = 0 \\ \varepsilon^2 :& \binom{3}{1} x_0 x_1^2 + \binom{3}{2} x_0^2 x_2 - 4x_2 - x_1 = 0 \end{aligned}$$

For $f_1(\varepsilon)$, $f_1(0) = x_0 = 0$, we can conclude consequently

$$x_0 = 0 \Rightarrow -4x_1 + 2 = 0 \Rightarrow x_1 = \frac{1}{2} \Rightarrow -4x_2 - \frac{1}{2} = 0 \Rightarrow x_2 = -\frac{1}{8}$$

$$f_1(\varepsilon) = 0 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \cdots, \quad x = f_1(0.001) \approx 0 + \frac{1}{2}(0.001) - \frac{1}{8}(0.001)^2 = 0.000499875$$

For $f_2(\varepsilon)$, $f_2(0) = x_0 = -2$, we can conclude consequently

$$x_0 = -2 \Rightarrow 12x_1 - 4x_1 + 2 + 2 = 0 \Rightarrow x_1 = -\frac{1}{2} \Rightarrow -\frac{3}{2} + 12x_2 - 4x_2 + \frac{1}{2} = 0 \Rightarrow x_2 = \frac{1}{8}$$

$$f_2(\varepsilon) = -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots, \quad x = f_2(0.001) \approx -2 - \frac{1}{2}(0.001) + \frac{1}{8}(0.001)^2 = -2.000499875$$

For $f_3(\varepsilon)$, $f_3(0) = x_0 = +2$, we can conclude consequently

$$x_0 = +2 \Rightarrow 12x_1 - 4x_1 - 2 + 2 = 0 \Rightarrow x_1 = 0 \Rightarrow 0 + 12x_2 - 4x_2 - 0 = 0 \Rightarrow x_2 = 0 \Rightarrow x_n = 0 (n \geq 3)$$

$$f_3(\varepsilon) = 2 + 0\varepsilon + 0\varepsilon^2 + \cdots = 2, \quad x = f_3(0.001) = 2$$

Finally we have 2 approximated roots $x \approx 0.000499875, -2.000499875$ and one exact root $x = 2$

```
>> syms x;
>> vpasolve(x^3 - 4.001*x + 0.002)
ans =
-2.0004998750624609648232582877001
0.00049987506246096482325828770010975
2.0
```

Problem 20

20. Find a two-term perturbation solution of

$$u' + u = \frac{1}{1+\varepsilon u}, \quad u(0) = 0, \quad 0 < \varepsilon \ll 1$$

Use software to plot the approximate solution with $\varepsilon = 0.1$ for $0 \le x \le 1$.
(Problem 20 on Page 169, PDF Page 201)

solution

We can find function $u = u(\varepsilon, x)$, and $u(0, x)$ is solution for $u' + u = \frac{1}{1+0} = 1, u(0) = 0$
Expand $u(\varepsilon, x)$ with ε

$$u(\varepsilon, x) = y_0(x) + y_1(x)\varepsilon + y_2(x)\varepsilon^2 + \cdots$$

Substitute $u(\varepsilon, x) = y_0(x) + y_1(x)\varepsilon + y_2(x)\varepsilon^2 + \cdots$ in $u' + u = \frac{1}{1+\varepsilon u}$
Compare the coefficients of $1, \varepsilon, \varepsilon^2$ respectively

$$\begin{aligned} 1 :& y_0' + y_0 = 1 \\ \varepsilon :& y_1' + y_1 = -y_0 \\ \varepsilon^2 :& y_2' + y_2 = -y_1 + y_0 \end{aligned}$$

Initial condition $u(\varepsilon, 0) = y_0(0) + y_1(0)\varepsilon + y_2(0)\varepsilon^2 + \cdots = 0$, compare the coefficients of $\varepsilon^k, k \in \mathrm{N}$

$$y_0(0) = y_1(0) = y_2(0) = \cdots = 0$$

Solve y_0

$$y_0 = e^{-\int 1dx}\left[\int 1 \cdot e^{\int 1dx} dx\right] = 1 + c_0 e^{-x}, y_0(0) = 0 \Rightarrow y_0 = 1 - e^{-x}$$

Solve y_1

$$y_1 = e^{-x}\left[\int -(1 - e^{-x}) \cdot e^x dx\right] = e^{-x}\left[-e^x + x + c_2\right] = -1 + xe^{-x} + c_1 e^{-x}, y_1(0) = 0 \Rightarrow y_1 = -1 + (x+1)e^{-x}$$

Solve y_2

$$\begin{aligned} y_2 &= e^{-x}\left[\int [-(-1 + (x+1)e^{-x}) + 1 - e^{-x}] \cdot e^x dx\right] = e^{-x}\left[\int [2 - (x+2)e^{-x}] \cdot e^x dx\right] \\ &= e^{-x}\left[2e^x - \left(\frac{1}{2}x^2 + 2x + c_2\right)\right], y_2(0) = 0 \\ &\Rightarrow y_1 = e^{-x}\left[2e^x - \left(\frac{1}{2}x^2 + 2x + 2\right)\right] = 2 - \left(\frac{1}{2}x^2 + 2x + 2\right)e^{-x} \end{aligned}$$

In all

$$\begin{aligned} y_0 &= 1 - e^{-x} \\ y_1 &= -1 + (x+1)e^{-x} \\ y_2 &= 2 - \left(\frac{1}{2}x^2 + 2x + 2\right)e^{-x} \\ u(\varepsilon, x) &\approx y_0(x) + y_1(x)\varepsilon + y_2(x)\varepsilon^2 \\ &= \left[1 - e^{-x}\right] + \varepsilon\left[-1 + (x+1)e^{-x}\right] + \varepsilon^2\left[2 - \left(\frac{1}{2}x^2 + 2x + 2\right)e^{-x}\right] \end{aligned}$$

The close form solution when $\varepsilon = 0.1$ for the separable equation, C is given by $u(0) = 0$

$$C - x = \frac{1}{14}\left[(7+\sqrt{35})\ln|-u+\sqrt{35}-5| - (\sqrt{35}-7)\ln|u+\sqrt{35}+5|\right]$$

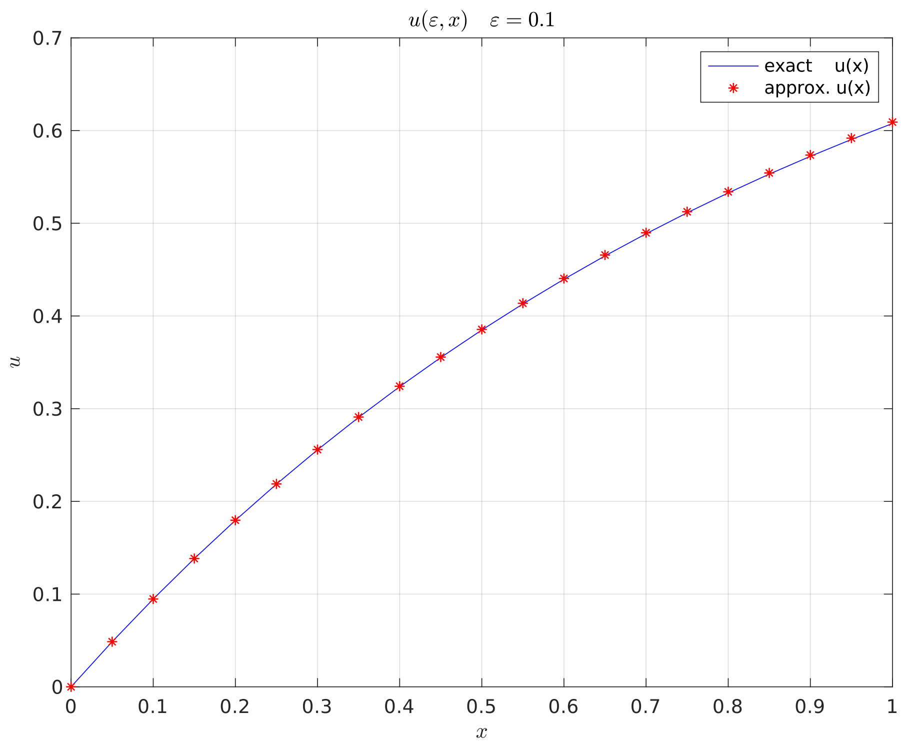


FIGURE 5. the exact $u(\varepsilon, x)$ and the approximate $u(\varepsilon, x)$ $(\varepsilon = 0.1)$

```
clear; clc; close all
% solve u(x) with ode23()
eps = 0.1;
func = @(x, u) 1 / (1 + eps*u) - u;
u0 = 0;
tspan = (0:0.05:1)';
[x,u] = ode23(func, tspan, u0);
plot(x,u,'b-'); xlabel('$x$','Interpreter','latex');
ylabel('$u$','Interpreter','latex');
title('$u(\varepsilon,x)\quad\varepsilon=0.1$','Interpreter','latex');
% calc approximate u(x)
y0 = @(x) 1 - exp(-x);
y1 = @(x) -1 + (x + 1).* exp(-x);
y2 = @(x) 2 - (x.^2 / 2 + 2 * x + 2).* exp(-x);
u_perturb = @(x) y0(x) + eps * y1(x) + eps^2 * y2(x);
u_approx = u_perturb(tspan);
hold on; plot(tspan, u_approx, 'r*'); grid on;
legend('exact    u(x)','approx. u(x)');
```

Bonus.

16. (Circuits) An RCL circuit with a nonlinear resistance, where the voltage drop across the resistor is a nonlinear function of current, can be modeled by the Van der Pol equation

$$x'' + \rho\left(x^2 - 1\right)x' + x = 0$$

where ρ is a positive constant, and $x(t)$ is the current.

a) In the phase plane, show that the origin is an unstable equilibrium.
b) Sketch the nullclines and the vector field. What are the possible dynamics? Is there a limit cycle?

(Problem 16ab on Page 111, PDF Page 134)

solution
With the substitution $y = x'$

$$\begin{aligned} x' &= y \\ y' &= x'' = -x - \rho(x^2-1)y \end{aligned}$$

The nullclines: $0 = y, 0 = -x - \rho(x^2-1)y$
the only critical point is $(0,0)$

For the direction in the separated regions:
region 1: $y > 0, -x - \rho(x^2-1)y > 0$
$(x', y') = (y, -x - \rho(x^2-1)y) = (+,+)$
region 2: $y < 0, -x - \rho(x^2-1)y > 0$
$(x', y') = (y, -x - \rho(x^2-1)y) = (-,+)$
region 3: $y < 0, -x - \rho(x^2-1)y < 0$
$(x', y') = (y, -x - \rho(x^2-1)y) = (-,-)$
region 4: $y > 0, -x - \rho(x^2-1)y < 0$
$(x', y') = (y, -x - \rho(x^2-1)y) = (+,-)$

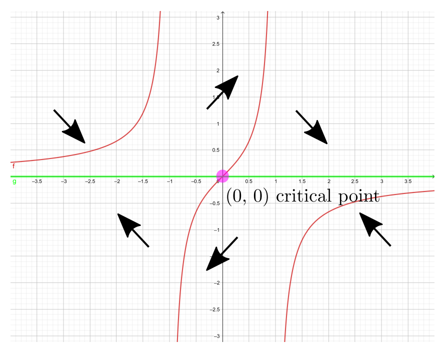


Figure 6. phase diagram $\rho = 1$

Consider the Jacobian J

$$J = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1-2\rho xy & -\rho(x^2-1) \end{pmatrix}$$

a) At the critical point $(0,0)$,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \rho \end{pmatrix}, \quad \det(J) \neq 0, \quad |J - \lambda I| = \lambda^2 - \rho\lambda + 1 = \lambda^2 - p\lambda + q = 0$$

With **Vieta theorem**, $\Delta = \rho^2 - 4, p = \lambda_1 + \lambda_2 = \mathrm{Re}(\lambda_1) + \mathrm{Re}(\lambda_2) = \rho > 0, q = 1$
if stable $\Rightarrow \mathrm{Re}(\lambda_1) \leq 0, \mathrm{Re}(\lambda_2) \leq 0 \Rightarrow \mathrm{Re}(\lambda_1) + \mathrm{Re}(\lambda_2) \leq 0$, there is a conflict, (0, 0) is **unstable**

b) The the nullclines and the vector field are displayed in the figure above

(1) $\Delta = \rho^2 - 4 = 0 \Rightarrow \rho = 2$
repeated eigenvalues $\lambda_1 = \lambda_2 = 1$, solution $c_1 w e^{\lambda_1 t} + c_2(w + vt)e^{\lambda_1 t}$
the critical point $(0,0)$ is an **unstable improper node**

(2) $\Delta = \rho^2 - 4 > 0 \Rightarrow 2 < \rho$
real eigenvalues $\lambda_1 > \lambda_2 > 0$
the critical point $(0,0)$ is an **unstable node**

(3) $\Delta = \rho^2 - 4 < 0 \Rightarrow 0 < \rho < 2$
complex eigenvalues $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = \frac{p}{2} = \frac{\rho}{2} > 0$
the critical point $(0,0)$ is an **unstable spiral**

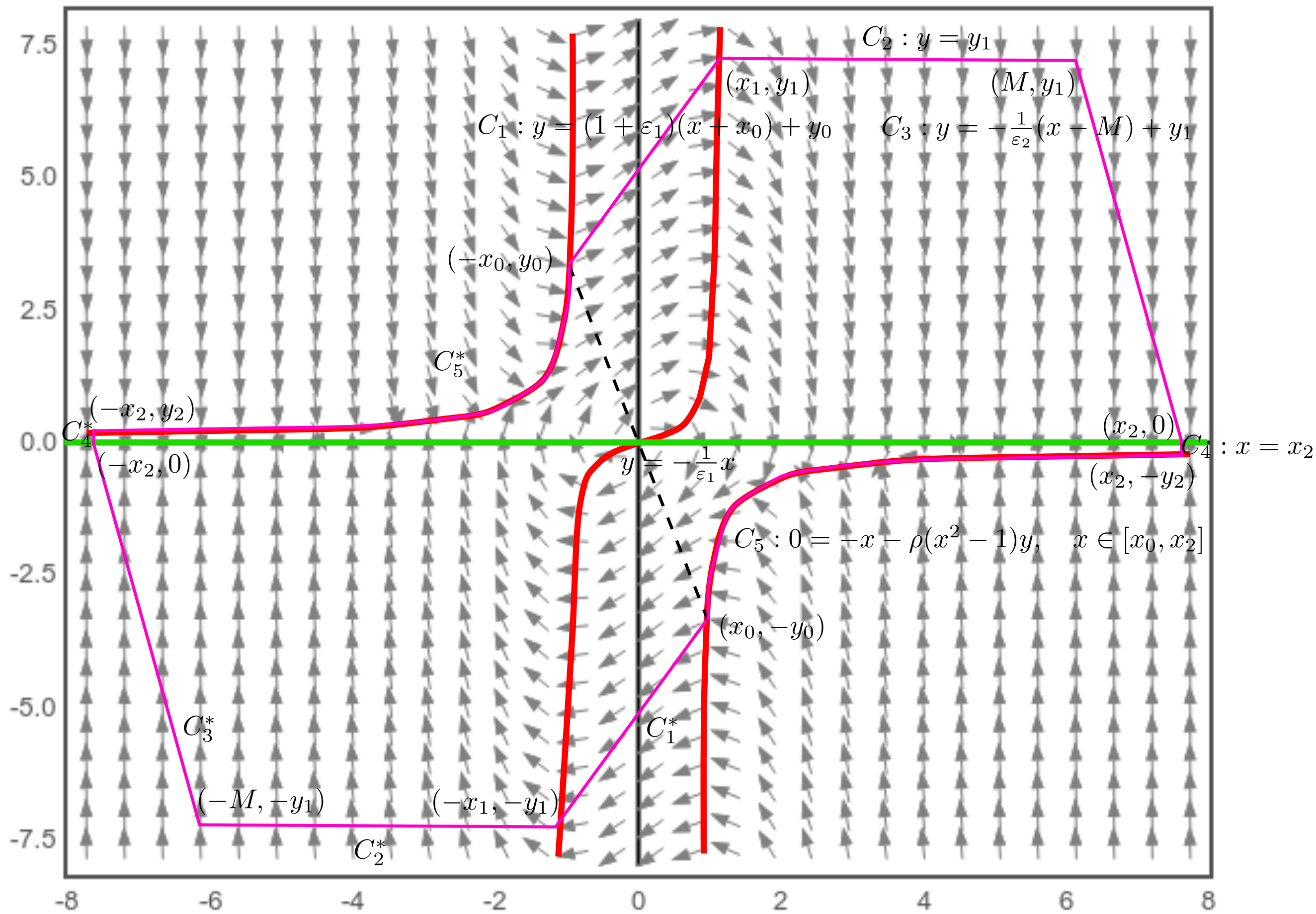


Figure 7. the simple closed curve $\boldsymbol{C}$, where $(x', y')^T \cdot \vec{n} < 0$

(Note: **Green**: x nullcline; **Red**: y nullcline; **Magenta**: the simple closed curve $\boldsymbol{C}$)

Theorem (Poincare-Bendixson Ring Domain Theorem). *Suppose* $\boldsymbol{R}$ *is the finite region of the plane lying between two simple closed curves* $\boldsymbol{C}$ *and* $\bar{\boldsymbol{C}}$, *and* $\boldsymbol{F}$ *is the velocity vector field for the system* $x' = f(x,y) \quad y' = g(x,y)$. *If*
(i) at each point of $\boldsymbol{C}$ *and* $\bar{\boldsymbol{C}}$, *the field* $\boldsymbol{F}$ *points toward the interior of* $\boldsymbol{R}$, *and*
(ii) $\boldsymbol{R}$ *contains no critical points,*
then the system has a ***closed trajectory*** *lying inside* $\boldsymbol{R}$

See the MIT limit cycle note or go to the url: `https://math.mit.edu/~jorloff/suppnotes/suppnotes03/lc.pdf`

For **Poincare-Bendixson Ring Domain Theorem**, we construct the simple closed curves first.
curve $\boldsymbol{C}$ consists of C_1, C_2, C_3, C_4, C_5 and the symmetrical curves $C_1^*, C_2^*, C_3^*, C_4^*, C_5^*$ of (0, 0)
For (x, y) on $\boldsymbol{C}$, always holds $(x', y')^T \cdot \vec{n} < 0$, here (x', y'): velocity field, $\vec{n}$: normal vector at (x, y)
Namely, for all (x, y) on $\boldsymbol{C}$, velocity field field (x', y') points toward the inside of $\boldsymbol{C}$
Short explanation for $(x', y')^T \cdot \vec{n} < 0$ on C_1, C_2, C_3, C_4, C_5 as follows:

(1) $C_1 : y = (\rho + \varepsilon_1)(x + x_0) + y_0$
where $(-x_0, y_0)$ is the intersection of $y = -\frac{1}{\varepsilon_1} x$ and left branch of $0 = -x - \rho(x^2 - 1)y$
ε_1 must satisfy, at $(-x_0, y_0)$: $\frac{dy}{dx}|_{(-x_0,y_0)}$ slope of $0 = -x - \rho(x^2 - 1)y > (\rho + \varepsilon_1)$ slope of C_1

$$(-x_0, y_0) = \left(-\sqrt{1 + \frac{\varepsilon_1}{\rho}}, \sqrt{1 + \frac{\varepsilon_1}{\rho}}/\varepsilon_1\right), \quad \text{left brach } 0 = -x - \rho(x^2 - 1)y \Leftrightarrow y = -\frac{1}{2\rho}\left(\frac{1}{x-1} + \frac{1}{x+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{\rho}\frac{x^2+1}{(x^2-1)^2}, \quad \frac{dy}{dx}|_{(-x_0,y_0)} = \frac{1}{\rho}\frac{2 + \frac{\varepsilon_1}{\rho}}{(\frac{\varepsilon_1}{\rho})^2} = \frac{2\rho + \varepsilon_1}{\varepsilon_1^2} > (\rho + \varepsilon_1) \Leftarrow \rho > 0 > -\varepsilon_1\left(\frac{1 - \varepsilon_1^2}{2 - \varepsilon_1^2}\right)$$

We can find $0 < \varepsilon_1 < 1$ to satisfy it, for $\forall \rho > 0$, it always holds

$$(x', y')^T = \left(y, -x - \rho(x^2 - 1)y\right)^T, \quad \vec{n} = (-\rho - \varepsilon_1, 1)^T$$

$$(x', y')^T \cdot \vec{n} = -\left[(\varepsilon_1 y + x) + \rho x^2\right] < 0$$

(2) $C_2 : y = y_1$
where (x_1, y_1) is the intersection of C_1 and center branch of $0 = -x - \rho(x^2 - 1)y$
for (x, y) on C_2, it has $x > x_1 > 0, y = y_1 > 0$, always holds

$$(x', y')^T = \left(y, -x - \rho(x^2 - 1)y\right)^T, \quad \vec{n} = (0, 1)^T$$

$$(x', y')^T \cdot \vec{n} = -x - \rho(x^2 - 1)y = \left[-x_1 - \rho(x_1^2 - 1)y_1\right] - (x - x_1) - (x^2 - x_1^2)y_1 = -(x - x_1) - (x^2 - x_1^2)y_1 < 0$$

(3) $C_3 : y = -\frac{1}{\varepsilon_2}(x - M) + y_1$
where (M, y_1) satisfies $M > \sqrt{1 + \frac{1}{\rho\varepsilon_2}}$
for (x, y) on C_3, it has $x > M, y > 0$, always holds

$$(x', y')^T = \left(y, -x - \rho(x^2 - 1)y\right)^T, \quad \vec{n} = (1, \varepsilon_2)^T$$

$$(x', y')^T \cdot \vec{n} = -\varepsilon_2 x - y\left[\rho\varepsilon_2(x^2 - 1) - 1\right] = -\varepsilon_2 x - y\left[\rho\varepsilon_2(M^2 - 1) - 1\right] - y\rho\varepsilon_2(x^2 - M^2) < 0$$

(4) $C_4 : x = x_2$
where $(x_2, 0)$ is the intersection of C_3 and $y = 0$
for (x, y) on C_4, it has $x = x_2, y < 0$, always holds

$$(x', y')^T = \left(y, -x - \rho(x^2 - 1)y\right)^T, \quad \vec{n} = (1, 0)^T$$

$$(x', y')^T \cdot \vec{n} = y < 0$$

(5) $C_5 : 0 = -x - \rho(x^2 - 1)y$ right branch
where $(x_2, -y_2)$ is the intersection of C_4 and $0 = -x - \rho(x^2 - 1)y$ right branch
for (x, y) on C_5, it has $x > x_0 = \sqrt{1 + \frac{\varepsilon_1}{\rho}} > 1, y < 0$, always holds

$$y = -\frac{1}{2\rho}\left(\frac{1}{x-1} + \frac{1}{x+1}\right), \quad \frac{dy}{dx} = \frac{1}{\rho}\frac{x^2+1}{(x^2-1)^2}$$

$$(x', y')^T = \left(y, -x - \rho(x^2 - 1)y\right)^T = (y, 0)^T, \quad \vec{n} = (\frac{dy}{dx}, -1)^T$$

$$(x', y')^T \cdot \vec{n} = -\left[\frac{1}{\rho}\frac{x^2+1}{(x^2-1)^2}\right](-y) < 0$$

To sum up, $(x', y')^T \cdot \vec{n} < 0$ holds on C_1, C_2, C_3, C_4, C_5, the velocity field is symmetrical to (0, 0). For the symmetrical curves $C_1^*, C_2^*, C_3^*, C_4^*, C_5^*$ of (0, 0), $(x', y')^T \cdot \vec{n} < 0$ holds on $C_1^*, C_2^*, C_3^*, C_4^*, C_5^*$. Thus, $(x', y')^T \cdot \vec{n} < 0$ holds for almost all points on entire $\boldsymbol{C}$
At each point of $\boldsymbol{C}$, the field $\boldsymbol{F} = (x', y')^T$ points toward the inside of $\boldsymbol{C}$
Now, consider to construct the simple closed curve $\bar{\boldsymbol{C}}$ (**Magenta**) near the critical point (0, 0)

(1) $\rho > 2, \Delta < 0$, construct $\bar{\boldsymbol{C}}$ as below
eigenvalues $\lambda_{1,2} = \frac{p \pm \sqrt{p^2-4}}{2} > 0$, with $(J - \lambda_{1,2})w_{1,2} = 0$, find $w_{1,2} = [1, \frac{p \pm \sqrt{p^2-4}}{2}]^T$
then $X(t) = (x, y)^T = c_1 e^{\lambda_1 t} w_1 + c_2 e^{\lambda_2 t} w_2$, $(x', y')^T = c_1 \lambda_1 e^{\lambda_1 t} w_1 + c_2 \lambda_2 e^{\lambda_2 t} w_2$
notice that coefficient $(c_1 \lambda_1 e^{\lambda_1 t}, c_2 \lambda_2 e^{\lambda_2 t})$ has the same signs as $(c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}), (c_1, c_2)$
the normal vector at (x, y) is $\vec{n} = \text{sgn}(c_1)\frac{w_1}{|w_1|} + \text{sgn}(c_2)\frac{w_2}{|w_2|}$, it always holds

$$
\begin{aligned}
(x', y')^T \cdot \vec{n} &= \left(c_1 \lambda_1 e^{\lambda_1 t} w_1 + c_2 \lambda_2 e^{\lambda_2 t} w_2\right) \cdot \left(\text{sgn}(c_1)\frac{w_1}{|w_1|} + \text{sgn}(c_2)\frac{w_2}{|w_2|}\right) \\
&= \left(c_1 \lambda_1 e^{\lambda_1 t}|w_1|\frac{w_1}{|w_1|} + c_2 \lambda_2 e^{\lambda_2 t}|w_2|\frac{w_2}{|w_2|}\right) \cdot \left(\text{sgn}(c_1 \lambda_1 e^{\lambda_1 t}|w_1|)\frac{w_1}{|w_1|} + \text{sgn}(c_2 \lambda_2 e^{\lambda_2 t}|w_2|)\frac{w_2}{|w_2|}\right) \\
&= (k_1 \vec{e}_1 + k_2 \vec{e}_2) \cdot (\text{sgn}(k_1)\vec{e}_1 + \text{sgn}(k_2)\vec{e}_2) \quad (\text{where } k_1 \equiv c_1 \lambda_1 e^{\lambda_1 t}|w_1|, \vec{e}_1 \equiv \frac{w_1}{|w_1|}) \\
&= |k_1| + |k_2| + (|k_1| + |k_2|)\,\text{sgn}(k_1 k_2)\vec{e}_1 \cdot \vec{e}_2 = (|k_1| + |k_2|)\,(1 + \text{sgn}(k_1 k_2)\vec{e}_1 \cdot \vec{e}_2) > 0
\end{aligned}
$$

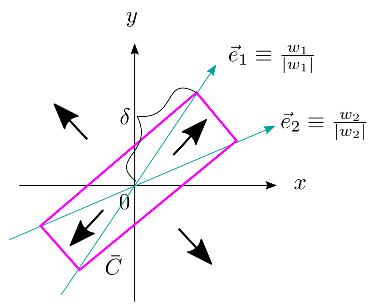


FIGURE 8. the simple closed curve $\bar{\boldsymbol{C}}$ (**Magenta**) near (0,0) for $\rho > 2$

(2) $0 < \rho \leq 2, \Delta \leq 0$, construct $\bar{\boldsymbol{C}}$ as below
it always holds $(x', y')^T \cdot \vec{n} > 0$

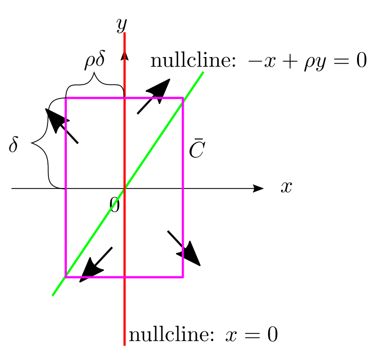


FIGURE 9. the simple closed curve $\bar{\boldsymbol{C}}$ (**Magenta**) near (0,0) for $0 < \rho \leq 2$

At each point of $\bar{\boldsymbol{C}}$, the field $\boldsymbol{F} = (x', y')^T$ points toward the outside of $\bar{\boldsymbol{C}}$
Conclusion: with **Poincare-Bendixson Ring Domain Theorem**,
there is a **closed trajectory** (**limit cycle**) inside $\boldsymbol{R}$ between two simple closed curves $\boldsymbol{C}$ and $\bar{\boldsymbol{C}}$

JOURNAL.

(100-300 words, please type.) Give a description of the regular perturbation method in your own words. Discuss the idea behind the method, the purpose of the method and the limitations of the method.

solution

Perturbation develops an expression for the desired solution in terms of a formal power series in some "small" parameter ε, namely a perturbation series that quantifies the deviation from the exactly solvable problem.

$$y(\varepsilon, x_1, x_2, \cdots) = \sum_{k=0}^{+\infty} y_k(x_1, x_2, \cdots)\varepsilon^k$$

$y_0(x_1, x_2, \cdots)$ is the known solution to the exactly solvable initial problem and $y_1(x_1, x_2, \cdots), y_2(x_1, x_2, \cdots), \cdots$ may be found iteratively by a mechanistic procedure. For small ε these higher-order terms in the series generally could become successively smaller.

Equations arising from mathematical models usually cannot be solved in exact form. The idea behind **perturbation** is that we breaks the problem into "solvable" and "perturbative" parts. Perturbation theory is widely used when the problem at hand does not have a known exact solution, but can be expressed as a "small" change to a known solvable problem. As a result, the computations of **perturbation** could be performed with a very high accuracy.

There are 2 limitations for regular perturbation

- The first one is called a **secular term**, like $t \sin t$
  In the approximation, the correction term $\sum_{k=N}^{\infty} y_k(t)\varepsilon^k$ cannot be made arbitrarily small for $t \in (0, +\infty)$ by choosing ε small enough.
  The solution is the **Poincaré-Lindstedt** Method, that is the a scale transformation, to avoid the presence of secular terms in the expansion.
- The other one is that regular perturbation assumed a leading term of order unity, and it is not surprising that it missed some roots. The roots are different order, and one expansion does not reveal both.
  The solution is **dominant balancing**, that means we examine each term carefully and determine which ones combine to give a dominant balance.