Problem 1.

1. Find the general solution and sketch phase diagrams for the following systems; characterize the equilibria as to type (node, etc.) and stability. (Problems 1abcd on Page 93, PDF Page 115)

a) $x' = x - 3y, y' = -3x + y$
b) $x' = -x + y, y' = y$
c) $x' = 4y, y' = -9x$
d) $x' = x + y, y' = 4x - 2y$

solution

a) The nullclines: $0 = x - 3y, 0 = -3x + y$, namely $y = \frac{1}{3}x, y = 3x$
The critical point is $(x, y) = (0, 0)$
The general solution is, when A is **diagonalizable**

$$X(t) = \exp(At)X(0) = \exp\left(W(\Lambda t)W^{-1}\right)X(0) = W\exp(\Lambda t)W^{-1}X(0)$$

Find eigenvalue $\Lambda = \left(\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right)$, and eigenvector $W = (w_1 \quad w_2)$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8 = 0$$

Here $\lambda_1 = 4, \lambda_2 = -2$,thus for w_1, w_2

$$(A - \lambda_1 I)w_1 = \begin{pmatrix} 1-\lambda_1 & -3 \\ -3 & 1-\lambda_1 \end{pmatrix} w_1 = \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix} w_1 = 0, \quad w_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A - \lambda_2 I)w_2 = \begin{pmatrix} 1-\lambda_2 & -3 \\ -3 & 1-\lambda_2 \end{pmatrix} w_2 = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} w_2 = 0, \quad w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Finally, where $[c_1 \quad c_2]^T \equiv W^{-1}X(0)$

$$X(t) = W\exp(\Lambda t)W^{-1}X(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \operatorname{diag}(e^{4t}, e^{-2t}) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

For the direction in the separated regions:
region 1: $y > \frac{1}{3}x, y > 3x$
$(x', y') = (x - 3y, -3x + y) = (-, +)$
region 2: $y < \frac{1}{3}x, y > 3x$
$(x', y') = (x - 3y, -3x + y) = (+, +)$
region 3: $y < \frac{1}{3}x, y < 3x$
$(x', y') = (x - 3y, -3x + y) = (+, -)$
region 4: $y > \frac{1}{3}x, y < 3x$
$(x', y') = (x - 3y, -3x + y) = (-, -)$

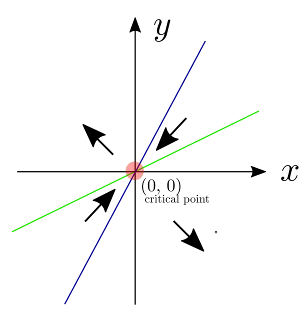


Figure 1. the phase diagram of a)

The equilibria (0, 0) has **saddle** structure, is **unstable**

b) The nullclines: $0 = -x + y, 0 = y$, namely $y = x, y = 0$
The critical point is $(x, y) = (0, 0)$
The general solution is, when A is **diagonalizable**

$$X(t) = \exp(At)X(0) = \exp\left(W(\Lambda t)W^{-1}\right)X(0) = W\exp(\Lambda t)W^{-1}X(0)$$

Find eigenvalue $\Lambda = \left(\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right)$, and eigenvector $W = (w_1 \quad w_2)$

$$|A - \lambda I| = \begin{vmatrix}-1-\lambda & 1\\ 0 & 1-\lambda\end{vmatrix} = \lambda^2 - 1 = 0$$

Here $\lambda_1 = 1, \lambda_2 = -1$,thus for w_1, w_2

$$(A - \lambda_1 I)w_1 = \begin{pmatrix}-1-\lambda_1 & 1\\ 0 & 1-\lambda_1\end{pmatrix} w_1 = \begin{pmatrix}-2 & 1\\ 0 & 0\end{pmatrix} w_1 = 0, \quad w_1 = \begin{bmatrix}1\\2\end{bmatrix}$$

$$(A - \lambda_2 I)w_2 = \begin{pmatrix}-1-\lambda_2 & 1\\ 0 & 1-\lambda_2\end{pmatrix} w_2 = \begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix} w_2 = 0, \quad w_2 = \begin{bmatrix}1\\0\end{bmatrix}$$

Finally, where $[c_1 \quad c_2]^T \equiv W^{-1}X(0)$

$$X(t) = W\exp(\Lambda t)W^{-1}X(0) = \begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix}\operatorname{diag}(e^t, e^{-t})\begin{bmatrix}c_1\\c_2\end{bmatrix} = c_1\begin{bmatrix}1\\2\end{bmatrix}e^t + c_2\begin{bmatrix}1\\0\end{bmatrix}e^{-t}$$

For the direction in the separated regions:
region 1: $y > x, y > 0$
$(x', y') = (-x + y, y) = (+, +)$
region 2: $y < x, y > 0$
$(x', y') = (-x + y, y) = (-, +)$
region 3: $y < x, y < 0$
$(x', y') = (-x + y, y) = (-, -)$
region 4: $y > x, y < 0$
$(x', y') = (-x + y, y) = (+, -)$

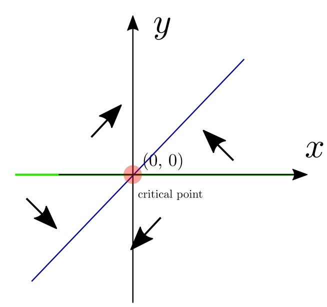


Figure 2. the phase diagram of b)

The equilibria (0, 0) has **saddle** structure, is **unstable**

c) The nullclines: $0 = 4y, 0 = -9x$, namely $y = 0, x = 0$
The critical point is $(x, y) = (0, 0)$
The general solution is, when A is **diagonalizable**

$$X(t) = \exp(At)X(0) = \exp\left(W(\Lambda t)W^{-1}\right)X(0) = W\exp(\Lambda t)W^{-1}X(0)$$

Find eigenvalue $\Lambda = \left(\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right)$, and eigenvector $W = (w_1 \quad w_2)$

$$|A - \lambda I| = \begin{vmatrix}-\lambda & 4\\ -9 & -\lambda\end{vmatrix} = \lambda^2 + 36 = 0$$

Here $\lambda_1 = 6i, \lambda_2 = -6i$,thus for w_1, w_2

$$(A - \lambda_1 I)w_1 = \begin{pmatrix}-\lambda_1 & 4\\ -9 & -\lambda_1\end{pmatrix} w_1 = \begin{pmatrix}-6i & 4\\ -9 & -6i\end{pmatrix} w_1 = 0, \quad w_1 = \begin{bmatrix}-2i\\ 3\end{bmatrix}$$

$$(A - \lambda_2 I)w_2 = \begin{pmatrix}-\lambda_2 & 4\\ -9 & -\lambda_2\end{pmatrix} w_2 = \begin{pmatrix}6i & 4\\ -9 & 6i\end{pmatrix} w_2 = 0, \quad w_2 = \begin{bmatrix}2i\\ 3\end{bmatrix}$$

Finally, where $\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}i\\ \frac{1}{2} & \frac{1}{2}i\end{pmatrix}[c_1 \quad c_2]^T \equiv W^{-1}X(0)$

$$\begin{aligned}
X(t) = W\exp(\Lambda t)W^{-1}X(0) &= \begin{pmatrix}-2i & 2i\\ 3 & 3\end{pmatrix} \operatorname{diag}(e^{6ti}, e^{-6ti}) \begin{pmatrix}\frac{1}{2} & -\frac{1}{2}i\\ \frac{1}{2} & \frac{1}{2}i\end{pmatrix} \begin{bmatrix}c_1\\ c_2\end{bmatrix}\\
&= \left[\operatorname{Re}\left(\begin{bmatrix}2i\\ 3\end{bmatrix} e^{6ti}\right) \quad \operatorname{Im}\left(\begin{bmatrix}2i\\ 3\end{bmatrix} e^{6ti}\right)\right] \begin{bmatrix}c_1\\ c_2\end{bmatrix}\\
&= c_1 \begin{bmatrix}-2\sin(6t)\\ 3\cos(6t)\end{bmatrix} + c_2 \begin{bmatrix}2\cos(6t)\\ 3\sin(6t)\end{bmatrix}
\end{aligned}$$

Where

$$\begin{bmatrix}2i\\ 3\end{bmatrix} e^{6ti} = \begin{bmatrix}i2\cos(6t) - 2\sin(6t)\\ 3\cos(6t) + i3\sin(6t)\end{bmatrix} = \begin{bmatrix}-2\sin(6t)\\ 3\cos(6t)\end{bmatrix} + i\begin{bmatrix}2\cos(6t)\\ 3\sin(6t)\end{bmatrix}$$

For the direction in the separated regions:
region 1: $y > 0, x > 0$
$(x', y') = (4y, -9x) = (+, -)$
region 2: $y > 0, x < 0$
$(x', y') = (4y, -9x) = (+, +)$
region 3: $y < 0, x < 0$
$(x', y') = (4y, -9x) = (-, +)$
region 4: $y < 0, x > 0$
$(x', y') = (4y, -9x) = (-, -)$

The equilibria (0, 0) has **center** structure, is **stable**

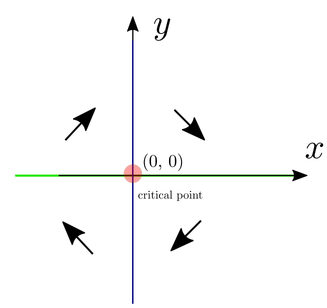


FIGURE 3. the phase diagram of c)

d) The nullclines: $0 = x + y, 0 = 4x - 2y$, namely $y = -x, y = 2x$
The critical point is $(x, y) = (0, 0)$
The general solution is, when A is **diagonalizable**

$$X(t) = \exp(At)X(0) = \exp\left(W(\Lambda t)W^{-1}\right)X(0) = W\exp(\Lambda t)W^{-1}X(0)$$

Find eigenvalue $\Lambda = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$, and eigenvector $W = (w_1 \quad w_2)$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = 0$$

Here $\lambda_1 = 2, \lambda_2 = -3$,thus for w_1, w_2

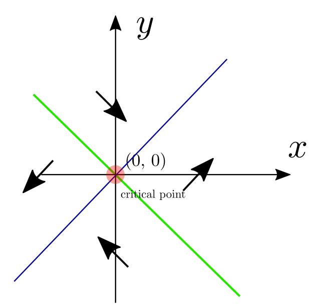
$$(A - \lambda_1 I)w_1 = \begin{pmatrix} 1-\lambda_1 & 1 \\ 4 & -2-\lambda_1 \end{pmatrix} w_1 = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} w_1 = 0, \quad w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)w_2 = \begin{pmatrix} 1-\lambda_2 & 1 \\ 4 & -2-\lambda_2 \end{pmatrix} w_2 = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} w_2 = 0, \quad w_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Finally, where $[c_1 \quad c_2]^T \equiv W^{-1}X(0)$

$$X(t) = W\exp(\Lambda t)W^{-1}X(0) = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \mathrm{diag}(e^{2t}, e^{-3t}) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$$

For the direction in the separated regions:
region 1: $y > -x, y > 2x$
$(x', y') = (x + y, 4x - 2y) = (+, -)$
region 2: $y < -x, y > 2x$
$(x', y') = (x + y, 4x - 2y) = (-, -)$
region 3: $y < -x, y < 2x$
$(x', y') = (x + y, 4x - 2y) = (-, +)$
region 4: $y > -x, y < 2x$
$(x', y') = (x + y, 4x - 2y) = (+, +)$



The equilibria (0, 0) has **saddle** structure, is **unstable**

Figure 4. the phase diagram of d)

Problem 1.

1. Determine the nature and stability properties of the critical points of the systems, and sketch the phase diagram: (Problems 1cde on Page 107, PDF Page 131)

c) $x' = y^2, y' = -2/3x$
d) $x' = x^2 - y^2, y' = x - y$
e) $x' = x^2 + y^2 - 4, y' = y - 2x$

solution
c) The nullclines: $0 = y^2, 0 = -2/3x$, the critical point is $(0, 0)$

For the direction in the separated regions:
region 1: $y^2 > 0, x > 0$
$(x', y') = (y^2, -2/3x) = (+, -)$
region 2: $y^2 > 0, x < 0$
$(x', y') = (y^2, -2/3x) = (+, +)$

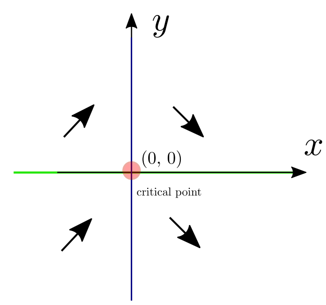


Figure 5. the phase diagram

Consider the Jacobian J

$$J = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 2y \\ -2/3 & 0 \end{pmatrix}$$

At the critical point $(0, 0)$,

$$J = \begin{pmatrix} 0 & 0 \\ -2/3 & 0 \end{pmatrix}, \quad \det(J) = 0$$

The equilibria (0, 0) has the degenerated type: **node**, is **unstable**

d) The nullclines: $0 = x^2 - y^2, 0 = x - y$, the critical point is $(x_0, x_0), x_0 \in \mathrm{R}$

For the direction in the separated regions:
region 1: $x + y > 0, x - y > 0$
$(x', y') = (x^2 - y^2, x - y) = (+, +)$
region 2: $x + y < 0, x - y > 0$
$(x', y') = (x^2 - y^2, x - y) = (-, +)$
region 3: $x + y < 0, x - y < 0$
$(x', y') = (x^2 - y^2, x - y) = (+, -)$
region 4: $x + y > 0, x - y < 0$
$(x', y') = (x^2 - y^2, x - y) = (-, -)$

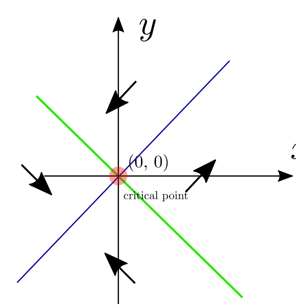


Figure 6. the phase diagram

Consider the Jacobian J

$$J = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 1 & -1 \end{pmatrix}$$

At the critical point (x_0, x_0),

$$J = \begin{pmatrix} 2x_0 & -2x_0 \\ 1 & -1 \end{pmatrix}, \quad \det(J) = 0, |J - \lambda I| = \lambda^2 - (-1 + 2x_0)\lambda = 0$$

The equilibria (x_0, x_0) has the degenerated type:**concentrated in a line x-y=0** (the line go through critical point) when $x_0 < 1/2$; **distracted from a line x-y=0** (the line go through critical point) when $x_0 > 1/2$;**node**, is **unstable** when $x_0 = 1/2$

e) The nullclines: $0 = x^2 + y^2 - 4, 0 = y - 2x$, the critical point is $(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}), (-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}})$

For the direction in the separated regions:
region 1: $x^2 + y^2 > 4, y > 2x$
$(x', y') = (x^2 + y^2 - 4, y - 2x) = (+, +)$
region 2: $x^2 + y^2 < 4, y > 2x$
$(x', y') = (x^2 + y^2 - 4, y - 2x) = (-, +)$
region 3: $x^2 + y^2 < 4, y < 2x$
$(x', y') = (x^2 + y^2 - 4, y - 2x) = (-, -)$
region 4: $x^2 + y^2 > 4, y < 2x$
$(x', y') = (x^2 + y^2 - 4, y - 2x) = (+, -)$

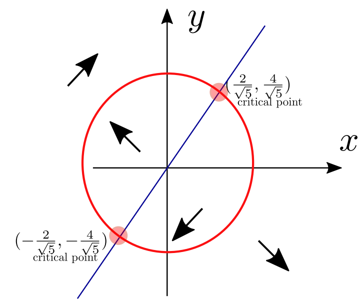


Figure 7. the phase diagram

Consider the Jacobian J

$$J = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ -2 & 1 \end{pmatrix}$$

At the critical point $(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}})$,

$$J = \begin{pmatrix} \frac{4}{\sqrt{5}} & \frac{8}{\sqrt{5}} \\ -2 & 1 \end{pmatrix}, \quad \det(J) \neq 0, \quad |J - \lambda I| = \lambda^2 + (-\frac{4}{\sqrt{5}} - 1)\lambda + \frac{20}{\sqrt{5}} = 0$$

With **Vieta theorem**, $\Delta = (\frac{4}{\sqrt{5}} + 1)^2 - 4 \times \frac{20}{\sqrt{5}} < 0, \lambda_1 + \lambda_2 = 2\text{Re}(\lambda_1) = 2\text{Re}(\lambda_1) = \frac{4}{\sqrt{5}} + 1 > 0$
The equilibria $(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}})$ has **spiral** structure, is **unstable**

At the critical point $(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}})$,

$$J = \begin{pmatrix} -\frac{4}{\sqrt{5}} & -\frac{8}{\sqrt{5}} \\ -2 & 1 \end{pmatrix}, \quad \det(J) \neq 0, \quad |J - \lambda I| = \lambda^2 + (\frac{4}{\sqrt{5}} - 1)\lambda - \frac{20}{\sqrt{5}} = 0$$

With **Vieta theorem**, $\Delta = (\frac{4}{\sqrt{5}} - 1)^2 + 4 \times \frac{20}{\sqrt{5}} > 0, \lambda_1\lambda_2 = -\frac{20}{\sqrt{5}} < 0$
Thus, $\lambda_1 > 0, \lambda_2 < 0$
The equilibria $(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}})$ has **saddle** structure, is **unstable**

Bonus.

1. Determine the nature and stability properties of the critical points of the systems, and sketch the phase diagram: (Problem 1a on Page 107, PDF Page 131)

a) $x' = x + y - 2x^2, y' = -2x + y + 3y^2$

solution

a) The nullclines: $0 = x + y - 2x^2, 0 = -2x + y + 3y^2$

Thus $x(12x^3 - 12x^2 + 5x - 3) = 0$, there is only one real root for $12x^3 - 12x^2 + 5x - 3 = 0$

It is $x = \frac{1}{6}\left(2 - \frac{1}{\sqrt[3]{20+\sqrt{401}}} + \sqrt[3]{20 + \sqrt{401}}\right) \approx 0.85472$

The critical points are $(0, 0), (0.85472, 0.606374)$

For the direction in the separated regions:

region 1: $y > 2x^2 - x, \frac{3}{2}y^2 + \frac{1}{2}y > x$

$(x', y') = (x + y - 2x^2, -2x + y + 3y^2) = (+, +)$

region 2: $y < 2x^2 - x, \frac{3}{2}y^2 + \frac{1}{2}y > x$

$(x', y') = (x + y - 2x^2, -2x + y + 3y^2) = (-, +)$

region 3: $y < 2x^2 - x, \frac{3}{2}y^2 + \frac{1}{2}y < x$

$(x', y') = (x + y - 2x^2, -2x + y + 3y^2) = (-, -)$

region 4: $y > 2x^2 - x, \frac{3}{2}y^2 + \frac{1}{2}y < x$

$(x', y') = (x + y - 2x^2, -2x + y + 3y^2) = (+, -)$

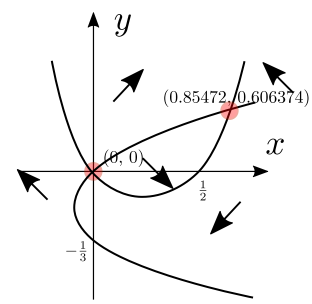


Figure 8. the phase diagram

Consider the Jacobian J

$$J = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 - 4x & 1 \\ -2 & 1 + 6y \end{pmatrix}$$

At the critical point $(0, 0)$,

$$J = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}, \quad \det(J) \neq 0, \quad |J - \lambda I| = \lambda^2 - 2\lambda + 3 = 0$$

With **Vieta theorem**, $\Delta = 2^2 - 4 \times 3 < 0, \lambda_1 + \lambda_2 = 2\text{Re}(\lambda_1) = 2\text{Re}(\lambda_2) = 2 > 0$

The equilibria (0, 0) has **spiral** structure, is **unstable**

At the critical point $(x, y) \approx (0.85472, 0.606374)$,

$$J \approx \begin{pmatrix} -2.41888 & 1 \\ -2 & 4.638244 \end{pmatrix}, \quad \det(J) \neq 0, \quad |J - \lambda I| \approx \lambda^2 - 2.219364\lambda - 9.21935564672 = 0$$

With **Vieta theorem**, $\Delta = 2.219364^2 + 4 \times 9.21935564672 > 0, \lambda_1\lambda_2 = -9.21935564672 < 0$

Thus, $\lambda_1 > 0, \lambda_2 < 0$

The equilibria (0, 0) has **saddle** structure, is **unstable**

Journal.

Write a paragraph that will help a student in MA 47200 to understand the concept of phase plane analysis and linearization method.

solution

The **phase plane analysis**:

The solutions to the differential equation are a family of functions. Graphically, this can be plotted in the phase plane like a two-dimensional vector field. Vectors representing the derivatives (x', y') with respect to a parameter t are drawn.

- It refers to graphically determining the existence of **limit cycles**:
  With enough of these arrows in place the system behaviour over the regions of plane can be visualized and limit cycles can be easily identified with **Poincaré-Bendixson theorem**.

- It is useful in determining if the critical points are **stable** or not:
  The eigenvalues (roots of $|A - \lambda I| = \lambda^2 - p\lambda + q = 0$) indicate the phase plane's behaviour:
  - Separated real eigenvalues ($\neq 0$) $\Delta > 0, q \neq 0$
    * If the signs are opposite, critical point is a **unstable saddle**.
    * If the signs are both positive, critical point is an **unstable node**.
    * If the signs are both negative, critical point is a **stable node**.
  - Complex eigenvalues $\Delta < 0, q \neq 0$
    * If the real part signs are both positive, critical point is an **unstable spiral**.
    * If the real part signs are both negative, critical point is a **stable spiral**.
    * If the real part signs are both 0, critical point is a **stable center**.
  - Repeated real eigenvalues ($\neq 0$) $\Delta = 0, q \neq 0$
    * the eigenvalue is positive $p > 0$, critical point is a **unstable node**.
      · $(A - \lambda I)w = 0$ eigenvector w_1, w_2: **proper node: star** $X = (c_1 w_1 + c_2 w_2)e^{\lambda t}$.
      · $(A - \lambda I)w = 0, (A - \lambda I)v = w$: **improper node** $X = c_1 w e^{\lambda t} + c_2(w + vt)e^{\lambda t}$.
    * the eigenvalue is negative $p < 0$, critical point is a **stable node**.
      · $(A - \lambda I)w = 0$ eigenvector w_1, w_2: **proper node: star** $X = (c_1 w_1 + c_2 w_2)e^{\lambda t}$.
      · $(A - \lambda I)w = 0, (A - \lambda I)v = w$: **improper node** $X = c_1 w e^{\lambda t} + c_2(w + vt)e^{\lambda t}$.
  - One of real eigenvalues $= 0$ $q = 0, p \neq 0$
    * the other eigenvalue is positive $p > 0$, critical point is a **unstable borderline**.
    * the other eigenvalue is negative $p < 0$, critical point is a **stable borderline**.

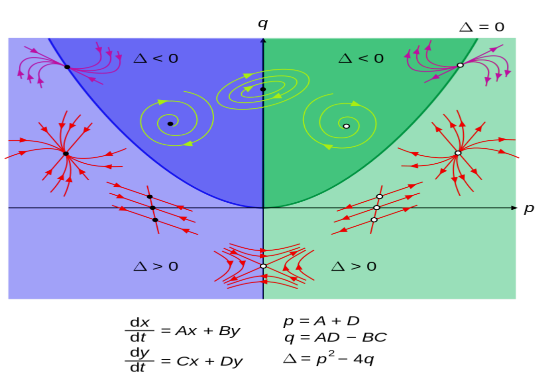


Figure 9. types of phase plane's behavior

The **linearization method**:
Linearization makes it possible to use tools for studying **linear systems** to analyze the behavior of a **nonlinear systems** near a given point. The linearization of a function is the **first order term of Taylor expansion** around the point of interest.
For an **autonomous systems** defined by the equation

$$d\begin{bmatrix} x \\ y \end{bmatrix}/dt = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

The critical point (x_0, y_0) satisfies

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} P(x_0,y_0) \\ Q(x_0,y_0) \end{bmatrix}$$

With first order term of Taylor expansion

$$d\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}/dt = d\begin{bmatrix} x \\ y \end{bmatrix}/dt = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} \approx \begin{bmatrix} P(x_0,y_0) \\ Q(x_0,y_0) \end{bmatrix} + \frac{\partial(P,Q)}{\partial(x,y)}|_{(x_0,y_0)} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} = J(x_0,y_0)\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

In **stability analysis** of autonomous systems, one can use the eigenvalues of the **Jacobian** $J(x_0, y_0)$ at the **critical point** (x_0, y_0) to determine the stability.