Problem 5.

5. (Harvesting) A deer population grows logistically, but it is harvested at a rate proportional to its population size. Conclude that the dynamics of population growth is given by

$$P' = rP\left(1 - \frac{P}{K}\right) - HP$$

where H is the per capita harvesting rate. Non-dimensionalize the model and use a bifurcation diagram to explain the effects on the equilibrium deer population when H is slowly increased from a small value.

solution

Non-dimensionalize the model

choose the characteristic scales $P_c = K, t_c = \frac{1}{r}$, define $p \equiv P/P_c, \tau \equiv t/t_c$

$$\frac{K}{\frac{1}{r}}\left(\frac{dp}{d\tau}\right) = rK\Big(p(1-p)\Big) - HKp \Longrightarrow \frac{dp}{d\tau} = p(1-p) - \frac{H}{r}p$$

Define the parameter $h \equiv \frac{H}{r}$

$$\frac{dp}{d\tau} = p(1-p) - hp = -p(p - [1-h]) \quad (p \geq 0) \tag{1}$$

$f(p)$ denotes the function on the right hand side. For the equilibria, $\frac{dp}{d\tau}|_{p=p^*} = 0$

$$0 = -p^*(p^* - [1-h]) \quad (p^* \geq 0)$$

condition 1: $0 \leq h < 1$
$p_1^* = 0$(unstable),
$p_2^* = 1 - h$(stable)
condition 3: $h = 1$
$p_1^* = 0$(semi-stable)
condition 3: $1 < h$
$p_1^* = 0$(stable)

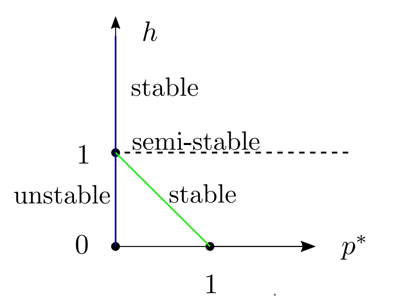


Figure 1. the bifurcation diagram

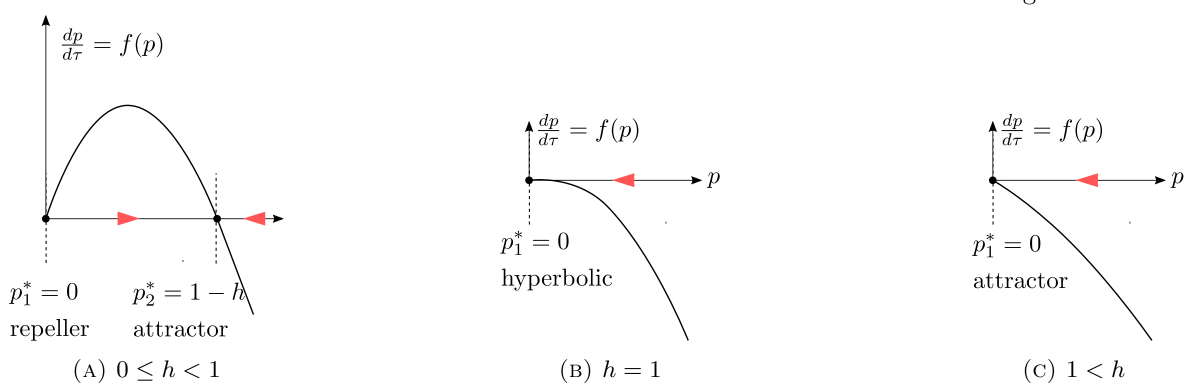


Figure 2. the diagram of p, $\frac{dp}{d\tau}$

Problem 6.

6. Draw a bifurcation diagram for the model $u' = u^3 - u + h$, where h is the bifurcation parameter, using h as the abscissa. Label branches of the curves as stable or unstable.
(Hint: first plot h vs. u.)

solution
$f(u) = u^3 - u + h$ denotes the function on the right hand side. For the equilibria, $\frac{du}{dt}|_{u=u^*} = 0$

$$0 = u^{*3} - u^* + h$$

The local minimal of $f(u)$: $(u_{\min}, f(u_{\min})) = (\frac{1}{\sqrt{3}}, -\frac{2}{3\sqrt{3}} + h)$
The local maximum of $f(u)$: $(u_{\max}, f(u_{\max})) = (-\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}} + h)$

condition 1: $h < -\frac{2}{3\sqrt{3}}$
u_1^*(unstable)
condition 3: $h = -\frac{2}{3\sqrt{3}}$
u_1^*(unstable),
u_2^*(semi-stable)
condition 3: $-\frac{2}{3\sqrt{3}} < h < \frac{2}{3\sqrt{3}}$
u_1^*(unstable),
u_2^*(stable),
u_3^*(unstable)
condition 4: $h = \frac{2}{3\sqrt{3}}$
u_2^*(semi-stable),
u_3^*(unstable)
condition 5: $\frac{2}{3\sqrt{3}} < h$
u_3^*(unstable)

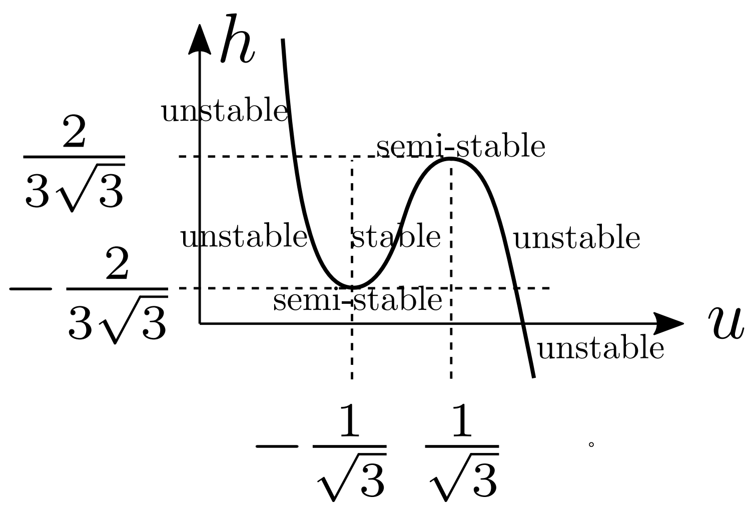
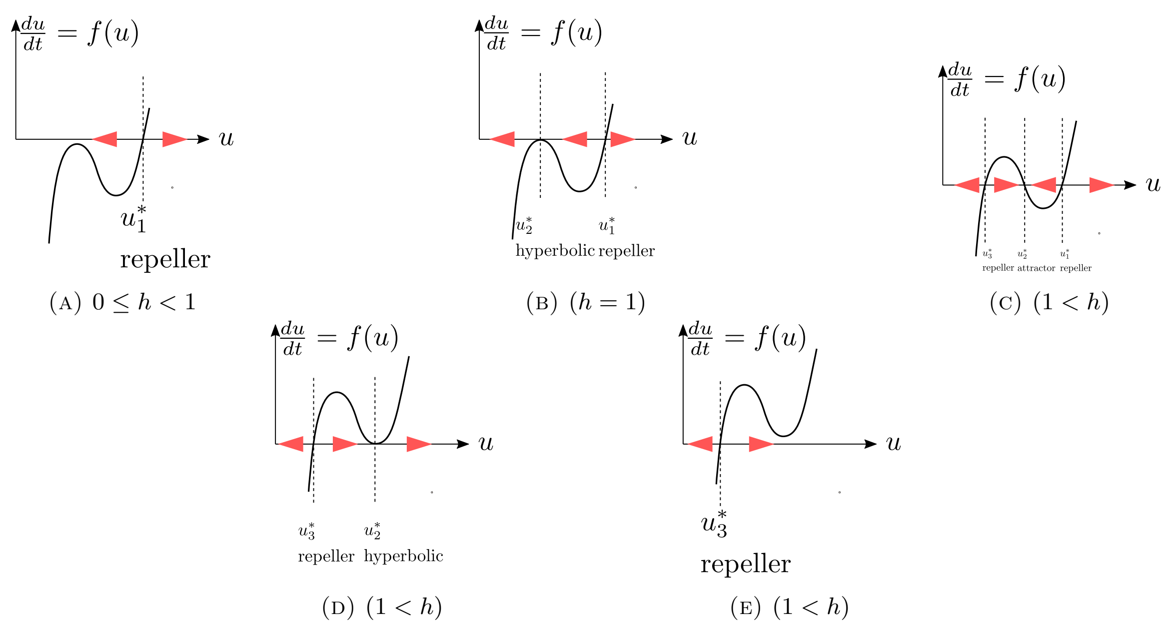


Figure 3. the bifurcation diagram



(A) $0 \leq h < 1$
(B) $(h = 1)$
(C) $(1 < h)$
(D) $(1 < h)$
(E) $(1 < h)$

Figure 4. the diagram of u, $\frac{du}{dt}$

Problem 8.

The following models contain a parameter h. Find the equilibria in terms of h and determine their stability. Construct a bifurcation diagram showing how equilibria depend upon h, and label the branches of the curves as unstable or stable.

a) $u' = hu - u^2$
b) $u' = hu - u^3$
c) $u' = (1-u)(u^2-h)$

solution
a) $f(u) = hu - u^2$ denotes the function on the right hand side. For the equilibria, $\frac{du}{dt}|_{u=u^*} = 0$

$$0 = hu^* - u^{*2} = -u(u-h)$$

condition 1: $h < 0$
$u_1^* = 0(\text{stable}), u_2^* = h(\text{unstable})$
condition 3: $h = 0$
$u_1^* = 0(\text{semi-stable})$
condition 3: $0 < h$
$u_1^* = 0(\text{unstable}), u_2^* = h(\text{stable})$

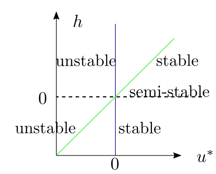


Figure 5. the bifurcation diagram

b) $f(u) = hu - u^3$ denotes the function on the right hand side. For the equilibria, $\frac{du}{dt}|_{u=u^*} = 0$

$$0 = hu^* - u^{*3} = -u(u^2-h)$$

condition 1: $h \leq 0$
$u_1^* = 0(\text{stable})$
condition 2: $0 < h$
$u_1^* = 0(\text{stable}), u_2^* = +\sqrt{h}(\text{unstable}), u_3^* = -\sqrt{h}(\text{stable})$

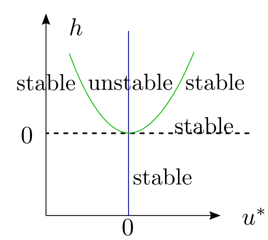


Figure 6. the bifurcation diagram

b) $f(u) = (1-u)(u^2-h)$ denotes the function on the right hand side. For the equilibria, $\frac{du}{dt}|_{u=u^*} = 0$

$$0 = (1-u)(u^2-h) = -(u-1)(u^2-h)$$

condition 1: $h \leq 0$
$u_1^* = 1(\text{stable})$
condition 2: $h = 0$
$u_1^* = 1(\text{stable}), u_2^* = 0(\text{semi-stable})$
condition 3: $0 < h < 1$
$u_1^* = 1(\text{stable}), u_2^* = +\sqrt{h}(\text{unstable}), u_3^* = -\sqrt{h}(\text{stable})$
condition 3: $0 < h < 1$
$u_1^* = 1(\text{semi-stable}), u_3^* = -\sqrt{h}(\text{stable})$
condition 5: $0 < h < 1$
$u_1^* = 1(\text{unstable}), u_2^* = +\sqrt{h}(\text{stable}), u_3^* = -\sqrt{h}(\text{stable})$

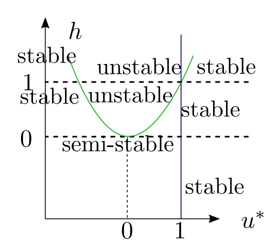


Figure 7. the bifurcation diagram

PROBLEM 3.

3. For the system

$$x' = y^2, y' = -\frac{2}{3}x$$

find the critical points, nullclines, and the direction of the orbits in the regions separated by the nullclines. Next, find the equations of the orbits and plot the phase plane diagram. Is the origin stable or unstable?

solution
The nullclines: $0 = y^2, 0 = -\frac{2}{3}x$, namely $y = 0, x = 0$
The critical point is $(x, y) = (0, 0)$
For the direction in the separated regions:

region 1: $x > 0, y > 0$
$(x', y') = (y^2, -\frac{2}{3}x) = (+, -)$
region 2: $x > 0, y < 0$
$(x', y') = (y^2, -\frac{2}{3}x) = (+, -)$
region 3: $x < 0, y > 0$
$(x', y') = (y^2, -\frac{2}{3}x) = (+, +)$
region 4: $x < 0, y < 0$
$(x', y') = (y^2, -\frac{2}{3}x) = (+, +)$

The origin is **unstable**

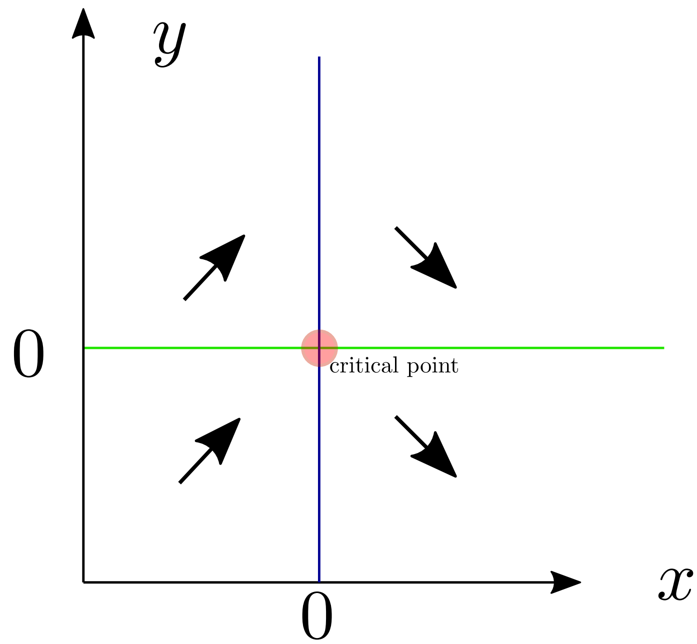


FIGURE 8. the direction of orbits

From equations

$$\frac{dy}{dx} = \frac{-\frac{2}{3}x}{y^2}$$

Separate, then integrate

$$\int y^2 dy = \int -\frac{2}{3}x dx$$

Thus

$$\frac{1}{3}y^3 = -\frac{1}{3}x^2 + c \Leftrightarrow y^3 + x^2 = C \tag{1}$$

Bonus.

10. (Ecology) An animal species of population u grows exponentially with growth rate r. At the same time it is subjected to predation at a rate $au/1+bu$, which depends upon its population. The constants a and b are positive, and the dynamics are

$$u' = ru - \frac{au}{1+bu}$$

a) Non-dimensionalize the model so that there is a single dimensionless parameter h.
b) Sketch a bifurcation diagram, and determine the stability of the equilibria as a function of h; indicate the results on the diagram.

solution

a) Non-dimensionalize the model

Select the characteristic scale $u_c = \frac{1}{b}, t_c = \frac{1}{r}$

Define $v \equiv u/u_c, \tau \equiv t/t_c$

$$\frac{r}{b}\left(\frac{dv}{d\tau}\right) = \frac{r}{b}v - \frac{a}{b}\left(\frac{v}{1+v}\right) \Longrightarrow \frac{dv}{d\tau} = v - \frac{a}{r}\left(\frac{v}{1+v}\right)$$

Define the parameter $h \equiv \frac{a}{r}$

$$\frac{dv}{d\tau} = v - h\left(\frac{v}{1+v}\right) = \frac{v\,(v-[h-1])}{v+1} \quad (v \geq 0)$$

b) Sketch a bifurcation diagram

$f(v)$ denotes the function on the right hand side. For the equilibria, $\frac{dv}{d\tau}|_{v=v^*} = 0$

$$0 = \frac{v^*\,(v^*-[h-1])}{v^*+1} \quad (v^* \geq 0)$$

condition 1: $0 \leq h < 1$
$p_1^* = 0$(unstable)
condition 2: $h = 1$
$p_1^* = 0$(semi-stable)
condition 3: $1 < h$
$p_1^* = 0$(stable),
$p_2^* = 1 + h$(unstable)

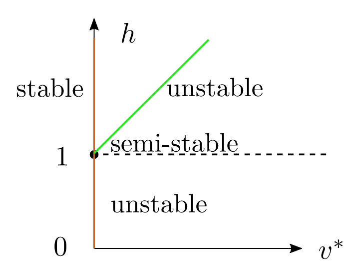
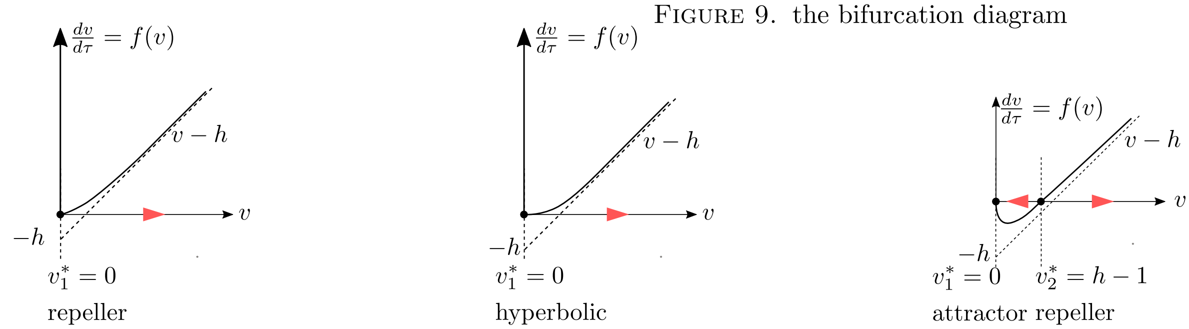


Figure 9. the bifurcation diagram



(a) $0 \leq h < 1$ (b) $h = 1$ (c) $1 < h$

Figure 10. the diagram of p, $\frac{dp}{dt}$

JOURNAL.

(100-300 words, please type.) Write a paragraph to explain the idea behind the phase line analysis and bifurcation phenomenon to a student in MA 472.

solution

A **phase line** is a diagram that shows the qualitative behaviour of an autonomous ordinary differential equation in a single variable $\frac{du}{dt} = f(u)$.

The phase line is the 1-dimensional form of the general phase space, and can be readily analyzed. The critical points (roots of the derivative $f(u) = 0$) are indicated, and the intervals between the critical points have their signs indicated with arrows. A critical point can be classified as **stable**, **unstable**, or **semi-stable**, by inspection of its neighbouring arrows.

A **bifurcation** occurs when a small smooth change made to the the **bifurcation parameters** of a system causes a sudden 'qualitative' or topological change in its behavior.

Particularly in dynamical systems, a **bifurcation diagram** shows the values visited or approached asymptotically (fixed points, periodic orbits, or chaotic attractors) of a system as a function of a bifurcation parameter in the system. Bifurcation diagrams enable the visualization of bifurcation theory.