Problem 1.

1. Find the general solution of the following differential equations:

(a) $u' + 2u = e^{-t}$
(c) $u' - tu = t^2u^2$
(e) $u'' + 9u = 3\sec 3t$
(l) $u'' + u' + u = te^t$

solution
(a) First order linear equation $u' + p(t)u = q(t)$, here $p(t) = 2$
Multiply integrating factor $\mu(t) \equiv e^{\int p(t)dt} = e^{\int 2dt} = e^{2t}$

$$\Big(\mu(t)u(t)\Big)' = \mu(t)e^{-t}$$

Integrate, then divide by $\mu(t)$

$$u(t) = \frac{1}{\mu(t)}\int \mu(t)e^{-t}dt = e^{-2t}[c_1 + e^t]$$

In the end

$$u(t) = c_1e^{-2t} + e^{-t} \tag{1}$$

(c) Bernoulli equation $u' + p(t)u = q(t)u^n$, here $n = 2$
Multiply $u^{-n} = u^{-2}$, let $v \equiv u^{1-n} = \frac{1}{u}$, $\frac{dv}{dt} = (1-n)u^{-n}u'$

$$\frac{1}{1-n}\frac{dv}{dt} - tv = u^{-n}u' - tu^{1-n} = t^2$$

Then simplify to $v' + p(t)u = q(t)$, here $p(t) = t$

$$\frac{dv}{dt} + tv = -t^2$$

Multiply integrating factor $\mu(t) \equiv e^{\int p(t)dt} = e^{\int tdt} = e^{\frac{1}{2}t^2}$

$$\Big(\mu(t)v(t)\Big)' = \mu(t)[-t^2]$$

Integrate, then divide by $\mu(t)$, note that $\mathrm{erfi}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{s^2}ds$

$$v(t) = \frac{1}{\mu(t)}\int \mu(t)[-t^2]dt = e^{-t^2/2}[c_1 - \int_0^t s^2e^{s^2/2}ds] = e^{-t^2/2}\Big[c_1 - te^{t^2/2} + \sqrt{\frac{\pi}{2}}\,\mathrm{erfi}\left(\frac{t}{\sqrt{2}}\right)\Big]$$

In the end, replace $u = v^{\frac{1}{1-n}} = \frac{1}{v}$

$$u(t) = \frac{e^{t^2/2}}{c_1 - te^{t^2/2} + \sqrt{\frac{\pi}{2}}\,\mathrm{erfi}\left(\frac{t}{\sqrt{2}}\right)} \tag{2}$$

(e) Non-homogeneous equation $u'' + a_1 u' + a_0 u = f(t)$, the general solution $u(t) = u_h(t) + u_p(t)$
For solution of homogeneous equation $u_h(t) = c_1 u_1(t) + c_2 u_2(t)$, the characteristic equation

$$r^2 + a_1 r + a_0 = r^2 + 9 = 0, \quad r_{1,2} = \pm 3i$$

To make sure u_1, u_2 are linear independent, select

$$u_1(t) \equiv \frac{e^{(+3i)t} - e^{(-3i)t}}{2i} = \sin(3t), \quad u_2(t) \equiv \frac{e^{(+3i)t} + e^{(-3i)t}}{2} = \cos(3t)$$

For the particular solution $u_p(t) = c_1(t)u_1(t) + c_2(t)u_2(t)$ with variation of parameters
Assume to satisfy the conditions ($n = 2$)

$$\sum_{i=1}^{n} c_i'(x) u_i^{(j)}(x) = 0, \quad j = 0, \ldots, n-2 \tag{a}$$

It leads to

$$u_p^{(j)}(t) = \begin{cases} \displaystyle\sum_{i=1}^{n} c_i(t) u_i^{(j)}(t), & j = 0, \ldots, n-1 \\ \displaystyle\sum_{i=1}^{n} c_i(t) u_i^{(n)}(t) + \sum_{i=1}^{n} c_i'(t) u_i^{(n-1)}(t), & j = n \end{cases}$$

The non-homogeneous equation becomes ($a_n = 1$)

$$\begin{aligned} \sum_{j=0}^{n} a_j u_p^{(j)}(t) &= \sum_{i=1}^{n} c_i'(t) u_i^{(n-1)}(t) + \sum_{i=1}^{n} c_i(t)[\sum_{j=0}^{n} a_j u_i^{(j)}(t)] \\ &= \sum_{i=1}^{n} c_i'(t) u_i^{(n-1)}(t) + \sum_{i=1}^{n} c_i(t) \times 0 = \sum_{i=1}^{n} c_i'(t) u_i^{(n-1)}(t) = f(t) \end{aligned} \tag{b}$$

Combine (a), (b)

$$\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

Wronskian $W(t) \equiv \left|\begin{smallmatrix} u_1 & u_2 \\ u_1' & u_2' \end{smallmatrix}\right|$ satisfies Abel's identity $W(t) = W(t_0)\exp(-\int_{t_0}^{t} a_{n-1} d\xi)$, here $t_0 = 0$

$$W(t) = \begin{vmatrix} \sin(3t) & \cos(3t) \\ 3\cos(3t) & -3\sin(3t) \end{vmatrix}_{t=0} \exp(-\int_0^t 0 d\xi) = -3$$

Use Cramer's rule to solve (a), (b)

$$\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} \left|\begin{smallmatrix} 0 & u_2 \\ f(t) & u_2' \end{smallmatrix}\right| / \left|\begin{smallmatrix} u_1 & u_2 \\ u_1' & u_2' \end{smallmatrix}\right| \\ \left|\begin{smallmatrix} u_1 & 0 \\ u_1' & f(t) \end{smallmatrix}\right| / \left|\begin{smallmatrix} u_1 & u_2 \\ u_1' & u_2' \end{smallmatrix}\right| \end{pmatrix} = \begin{pmatrix} -\frac{f(t)u_2(t)}{W(t)} \\ \frac{f(t)u_1(t)}{W(t)} \end{pmatrix}$$

Integrate $c_1(t), c_2(t)$, select constants of integration $= 0$ for the particular solution $u_p(t)$

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} -\int \frac{f(t)u_2(t)}{W(t)} dt \\ \int \frac{f(t)u_1(t)}{W(t)} dt \end{pmatrix} = \begin{pmatrix} -\int \frac{3\sec(3t)\cos(3t)}{-3} dt \\ \int \frac{3\sec(3t)\sin(3t)}{-3} dt \end{pmatrix} = \begin{pmatrix} \int 1 dt \\ -\int \tan(3t) dt \end{pmatrix} = \begin{pmatrix} t \\ \frac{1}{3}\ln(\cos(3t)) \end{pmatrix}$$

$$u_p(t) = c_1(t)u_1(t) + c_2(t)u_2(t) = t\sin(3t) + \frac{1}{3}\cos(3t)\ln(\cos(3t))$$

In the end, the general solution $u(t) = u_h(t) + u_p(t)$

$$u(t) = c_1(t)\sin(3t) + c_2(t)\cos(3t) + t\sin(3t) + \frac{1}{3}\cos(3t)\ln(\cos(3t)) \tag{3}$$

(l) Non-homogeneous equation $u'' + a_1 u' + a_0 u = f(t)$, the general solution $u(t) = u_h(t) + u_p(t)$
For solution of homogeneous equation $u_h(t) = c_1 u_1(t) + c_2 u_2(t)$, the characteristic equation

$$r^2 + a_1 r + a_0 = r^2 + r + 1 = 0, \quad r_{1,2} = \frac{-1 \pm \sqrt{3}i}{2}$$

To make sure u_1, u_2 are linear independent, select

$$u_1(t) \equiv \frac{e^{(\frac{-1+\sqrt{3}i}{2})t} - e^{(\frac{-1-\sqrt{3}i}{2})t}}{2i} = e^{-t/2}\sin(\frac{\sqrt{3}}{2}t), \quad u_2(t) \equiv \frac{e^{(\frac{-1+\sqrt{3}i}{2})t} + e^{(\frac{-1-\sqrt{3}i}{2})t}}{2} = e^{-t/2}\cos(\frac{\sqrt{3}}{2}t)$$

Method 1: variation of parameters
For the particular solution $u_p(t) = c_1(t)u_1(t) + c_2(t)u_2(t)$ with variation of parameters
Assume to satisfy the conditions ($n = 2$)

$$\sum_{i=1}^{n} c_i'(x) u_i^{(j)}(x) = 0, \quad j = 0, \ldots, n-2 \tag{a}$$

It leads to

$$u_p^{(j)}(t) = \begin{cases} \sum_{i=1}^{n} c_i(t) u_i^{(j)}(t), & j = 0, \ldots, n-1 \\ \sum_{i=1}^{n} c_i(t) u_i^{(n)}(t) + \sum_{i=1}^{n} c_i'(t) u_i^{(n-1)}(t), & j = n \end{cases}$$

The non-homogeneous equation becomes ($a_n = 1$)

$$\begin{aligned} \sum_{j=0}^{n} a_j u_p^{(j)}(t) &= \sum_{i=1}^{n} c_i'(t) u_i^{(n-1)}(t) + \sum_{i=1}^{n} c_i(t)[\sum_{j=0}^{n} a_j u_i^{(j)}(t)] \\ &= \sum_{i=1}^{n} c_i'(t) u_i^{(n-1)}(t) + \sum_{i=1}^{n} c_i(t) \times 0 = \sum_{i=1}^{n} c_i'(t) u_i^{(n-1)}(t) = f(t) \end{aligned} \tag{b}$$

Combine (a), (b)

$$\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

Wronskian $W(t) \equiv |\begin{smallmatrix} u_1 & u_2 \\ u_1' & u_2' \end{smallmatrix}|$ satisfies Abel's identity $W(t) = W(t_0)\exp(-\int_{t_0}^{t} a_{n-1} d\xi)$, here $t_0 = 0$

$$W(t) = \begin{vmatrix} e^{-t/2}\sin(\frac{\sqrt{3}}{2}t) & e^{-t/2}\cos(\frac{\sqrt{3}}{2}t) \\ e^{-t/2}[-\frac{1}{2}\sin(\frac{\sqrt{3}}{2}t) + \frac{\sqrt{3}}{2}\cos(\frac{\sqrt{3}}{2}t)] & e^{-t/2}[-\frac{\sqrt{3}}{2}\sin(\frac{\sqrt{3}}{2}t) - \frac{1}{2}\cos(\frac{\sqrt{3}}{2}t)] \end{vmatrix}_{t=0} \exp(-\int_0^t 1 d\xi) = -\frac{\sqrt{3}}{2}e^{-t}$$

Use Cramer's rule to solve (a), (b)

$$\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} |\begin{smallmatrix} 0 & u_2 \\ f(t) & u_2' \end{smallmatrix}| / |\begin{smallmatrix} u_1 & u_2 \\ u_1' & u_2' \end{smallmatrix}| \\ |\begin{smallmatrix} u_1 & 0 \\ u_1' & f(t) \end{smallmatrix}| / |\begin{smallmatrix} u_1 & u_2 \\ u_1' & u_2' \end{smallmatrix}| \end{pmatrix} = \begin{pmatrix} -\frac{f(t)u_2(t)}{W(t)} \\ \frac{f(t)u_1(t)}{W(t)} \end{pmatrix}$$

Integrate $c_1(t), c_2(t)$, select constants of integration $= 0$ for the particular solution $u_p(t)$

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} -\int \frac{f(t)u_2(t)}{W(t)} dt \\ \int \frac{f(t)u_1(t)}{W(t)} dt \end{pmatrix} = \begin{pmatrix} -\int \frac{te^t e^{-t/2}\cos(\frac{\sqrt{3}}{2}t)}{-\frac{\sqrt{3}}{2}e^{-t}} dt \\ \int \frac{te^t e^{-t/2}\sin(\frac{\sqrt{3}}{2}t)}{-\frac{\sqrt{3}}{2}e^{-t}} dt \end{pmatrix} = \frac{2}{\sqrt{3}} \begin{pmatrix} \int te^{\frac{3}{2}t}\cos(\frac{\sqrt{3}}{2}t) dt \\ -\int te^{\frac{3}{2}t}\sin(\frac{\sqrt{3}}{2}t) dt \end{pmatrix}$$

$$= \frac{1}{3\sqrt{3}} e^{\frac{3}{2}t} \begin{pmatrix} \left(\sqrt{3}(t-1)\sin\left(\frac{\sqrt{3}t}{2}\right) + (3t-1)\cos\left(\frac{\sqrt{3}t}{2}\right)\right) \\ \left((1-3t)\sin\left(\frac{\sqrt{3}t}{2}\right) + \sqrt{3}(t-1)\cos\left(\frac{\sqrt{3}t}{2}\right)\right) \end{pmatrix}$$

$$u_p(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}^T \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = e^{-t/2} \begin{pmatrix} \sin(\frac{\sqrt{3}}{2}t) \\ \cos(\frac{\sqrt{3}}{2}t) \end{pmatrix}^T \frac{1}{3\sqrt{3}} e^{\frac{3}{2}t} \begin{pmatrix} \left(\sqrt{3}(t-1)\sin\left(\frac{\sqrt{3}t}{2}\right) + (3t-1)\cos\left(\frac{\sqrt{3}t}{2}\right)\right) \\ \left((1-3t)\sin\left(\frac{\sqrt{3}t}{2}\right) + \sqrt{3}(t-1)\cos\left(\frac{\sqrt{3}t}{2}\right)\right) \end{pmatrix}$$

$$= \frac{1}{3\sqrt{3}} e^t \sqrt{3}(t-1) = \frac{te^t}{3} - \frac{e^t}{3}$$

In the end, the general solution $u(t) = u_h(t) + u_p(t)$

$$u(t) = c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{te^t}{3} - \frac{e^t}{3} \tag{4}$$

Method 2: undetermined coefficients

Because $f(t) = te^t$ is a product of special functions t, e^t, we can set the particular solution $u_p(t)$

$$u_p(t) = (At + B)e^t$$

It satisfies

$$u_p'' + u_p' + u_p = (At + 2A + B)e^t + (At + A + B)e^t + (At + B)e^t = [(3A)t + (3A + 3B)]\, e^t = te^t$$

Compare the coefficients of te^t, e^t

$$\begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$u_p(t) = \frac{te^t}{3} - \frac{e^t}{3}$$

In the end, we can obtain the general solution $u(t) = u_h(t) + u_p(t)$ as well as (4)

$$u(t) = c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{te^t}{3} - \frac{e^t}{3}$$

Problem 4.

4. (Reactor dynamics) Consider a chemical reactor of constant volume V where a chemical C is pumped into the reactor at constant concentration and constant flow rate q. While in the reactor it reacts according to C + C $\rightarrow$ products. The law of mass action dictates that the rate of the reaction is $r = kC^2$, where k is the rate constant. If the concentration of C in the reactor is given by $C(t)$, then mass balance leads to the governing equation

$$(VC)' = qc_{\text{in}} - qC - kVC^2, \quad C(0) = c_0$$

Non-dimensionalize the model by selecting the concentration scale to be c_{in}, the input concentration, and a time scale based on the flow-through rate. Determine the equilibria, or constant solutions, and find a formula for the concentration as a function of time.

solution

Select the concentration scale c_{in}, and time scale $t_c \equiv \frac{V}{q}$ based on the flow-through rate
Non-dimensionalize the model by replacing C, t with $\bar{C} \equiv C/c_{\text{in}}, \bar{t} \equiv t/(\frac{V}{q})$

$$\frac{c_{\text{in}}}{(\frac{V}{q})}\left[\frac{d\bar{C}}{d\bar{t}}\right] = c_{\text{in}}\frac{q}{V} - c_{\text{in}}\frac{q}{V}\bar{C} - c_{\text{in}}^2\left[k\bar{C}^2\right], \quad c_{\text{in}}\bar{C}(0) = c_0$$

$$\frac{d\bar{C}}{d\bar{t}} = 1 - \bar{C} - kc_{\text{in}}\frac{V}{q}\bar{C}^2 \quad \bar{C}(0) = \frac{c_0}{c_{\text{in}}}$$

Define $\epsilon \equiv kc_{\text{in}}\frac{V}{q}, \alpha = \frac{c_0}{c_{\text{in}}}$, it becomes

$$\frac{d\bar{C}}{d\bar{t}} = 1 - \bar{C} - \epsilon\bar{C}^2 \quad \bar{C}(0) = \alpha$$

Determine the equilibria $C_{\text{eq}} = \bar{C}_{\text{eq}}c_{\text{in}}$ by setting $\frac{d\bar{C}}{d\bar{t}} = 0$

$$0 = 1 - \bar{C}_{\text{eq}} - \epsilon\bar{C}_{\text{eq}}^2, \quad \bar{C}_{\text{eq}} > 0 \Longrightarrow \bar{C}_{\text{eq}} = \frac{2}{1 + \sqrt{1+4\epsilon}} = \frac{\sqrt{1+4\epsilon} - 1}{2\epsilon}$$

The the equilibria C_{eq} is

$$C_{\text{eq}} = \bar{C}_{\text{eq}}c_{\text{in}} = \frac{\sqrt{1+4\epsilon} - 1}{2\epsilon}c_{\text{in}} = \frac{\sqrt{1 + 4kc_{\text{in}}\frac{V}{q}} - 1}{2k\frac{V}{q}} \tag{1}$$

Method 1: integrate the separable equation

If the initial value $c_0 = C_{\text{eq}} = \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}} - 1}{2k\frac{V}{q}}$

$$c(t) = C_{\text{eq}} = \frac{\sqrt{1 + 4kc_{\text{in}}\frac{V}{q}} - 1}{2k\frac{V}{q}} \tag{a}$$

Otherwise, separate $\bar{C}, \bar{t}$, then integrate

$$\int \frac{d\bar{C}}{1 - \bar{C} - \epsilon\bar{C}^2} = \frac{\ln\left|\frac{\bar{C} + \frac{\sqrt{1+4\epsilon}+1}{2\epsilon}}{\bar{C} + \frac{-\sqrt{1+4\epsilon}+1}{2\epsilon}}\right|}{\sqrt{1+4\epsilon}} = \frac{\ln\left|\frac{\frac{2\epsilon\bar{C}+1}{\sqrt{1+4\epsilon}} + 1}{\frac{2\epsilon\bar{C}+1}{\sqrt{1+4\epsilon}} - 1}\right|}{\sqrt{1+4\epsilon}} = \int d\bar{t} = \bar{t} + c_1$$

Notice that

$$\frac{\ln\left|\frac{\frac{2\epsilon\bar{C}+1}{\sqrt{1+4\epsilon}}+1}{\frac{2\epsilon\bar{C}+1}{\sqrt{1+4\epsilon}}-1}\right|}{\sqrt{1+4\epsilon}} = \begin{cases} \frac{2\tanh^{-1}(\frac{2\epsilon\bar{C}+1}{\sqrt{1+4\epsilon}})}{\sqrt{1+4\epsilon}} & \frac{2\epsilon\bar{C}+1}{\sqrt{1+4\epsilon}} < 1 \Leftrightarrow \bar{C} < \frac{\sqrt{1+4\epsilon}-1}{2\epsilon} = \bar{C}_{\text{eq}} \\ \frac{2\coth^{-1}(\frac{2\epsilon\bar{C}+1}{\sqrt{1+4\epsilon}})}{\sqrt{1+4\epsilon}} & \frac{2\epsilon\bar{C}+1}{\sqrt{1+4\epsilon}} > 1 \Leftrightarrow \bar{C} > \frac{\sqrt{1+4\epsilon}-1}{2\epsilon} = \bar{C}_{\text{eq}} \end{cases}$$

If the initial value $c_0 < C_{\text{eq}} = \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}-1}{2k\frac{V}{q}}$, where $c_1 = \frac{2\tanh^{-1}(\frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}})}{\sqrt{1+4\epsilon}} = \frac{2\tanh^{-1}(\frac{2kc_0(V/q)+1}{\sqrt{1+4kc_{\text{in}}(V/q)}})}{\sqrt{1+4kc_{\text{in}}(V/q)}}$

$$c(t) = \bar{C}(\bar{t})c_{\text{in}} = \frac{\sqrt{1+4\epsilon}\tanh\left(\frac{\sqrt{1+4\epsilon}}{2}(\bar{t}+c_1)\right)-1}{2\epsilon}c_{\text{in}} = \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}\tanh\left(\frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}}{2}\left(\frac{q}{V}t+c_1\right)\right)-1}{2k\frac{V}{q}} \tag{b}$$

If the initial value $c_0 > C_{\text{eq}} = \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}-1}{2k\frac{V}{q}}$, where $c_1 = \frac{2\coth^{-1}(\frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}})}{\sqrt{1+4\epsilon}} = \frac{2\coth^{-1}(\frac{2kc_0(V/q)+1}{\sqrt{1+4kc_{\text{in}}(V/q)}})}{\sqrt{1+4kc_{\text{in}}(V/q)}}$

$$c(t) = \bar{C}(\bar{t})c_{\text{in}} = \frac{\sqrt{1+4\epsilon}\coth\left(\frac{\sqrt{1+4\epsilon}}{2}(\bar{t}+c_1)\right)-1}{2\epsilon}c_{\text{in}} = \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}\coth\left(\frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}}{2}\left(\frac{q}{V}t+c_1\right)\right)-1}{2k\frac{V}{q}} \tag{c}$$

To sum up

$$c(t) = \begin{cases} \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}-1}{2k\frac{V}{q}} & \text{for } c_0 = \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}-1}{2k\frac{V}{q}} \\ \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}\tanh\left(\frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}}{2}\left(\frac{q}{V}t+\frac{2\tanh^{-1}(\frac{2kc_0(V/q)+1}{\sqrt{1+4kc_{\text{in}}(V/q)}})}{\sqrt{1+4kc_{\text{in}}(V/q)}}\right)\right)-1}{2k\frac{V}{q}} & \text{for } c_0 < \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}-1}{2k\frac{V}{q}} \\ \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}\coth\left(\frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}}{2}\left(\frac{q}{V}t+\frac{2\coth^{-1}(\frac{2kc_0(V/q)+1}{\sqrt{1+4kc_{\text{in}}(V/q)}})}{\sqrt{1+4kc_{\text{in}}(V/q)}}\right)\right)-1}{2k\frac{V}{q}} & \text{for } c_0 > \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}-1}{2k\frac{V}{q}} \end{cases} \tag{2}$$

Method 2: solve the Riccati equation

Riccati equations $\frac{d\bar{C}}{d\bar{t}} + \bar{C} = -\epsilon\bar{C}^2 + 1$

The general solution $\bar{C}(\bar{t}) = \bar{C}_h(\bar{t}) + \bar{C}_p(\bar{t})$, where a particular solution $\bar{C}_p(\bar{t}) = \bar{C}_{\text{eq}} = \frac{\sqrt{1+4\epsilon}-1}{2\epsilon}$

And the solution $\bar{C}_h(\bar{t})$ for a Bernoulli equation (n=2)

$$\frac{d\bar{C}_h}{d\bar{t}} + \left[1 + 2\epsilon\frac{\sqrt{1+4\epsilon}-1}{2\epsilon}\right]\bar{C}_h = \frac{d\bar{C}_h}{d\bar{t}} + \sqrt{1+4\epsilon}\bar{C}_h = -\epsilon\bar{C}_h^2$$

With substitution $v \equiv \bar{C}_h^{1-n} = \frac{1}{\bar{C}_h}$, here assume $\alpha \neq \bar{C}_{\text{eq}}$

$$\frac{dv}{d\bar{t}} - \left(\sqrt{1+4\epsilon}\right) v = \epsilon, \quad v(0) = \frac{1}{\alpha - \bar{C}_{\text{eq}}} = \frac{1}{\alpha - \left[\frac{\sqrt{1+4\epsilon}-1}{2\epsilon}\right]}$$

Multiply the integrating factor $\mu(\bar{t}) = e^{\int p(\bar{t})d\bar{t}} = e^{\int -\sqrt{1+4\epsilon}d\bar{t}} = e^{-\sqrt{1+4\epsilon}\bar{t}}$

$$v(\bar{t}) = \frac{1}{\mu(\bar{t})}\int \epsilon\mu(\bar{t})d\bar{t} = e^{\sqrt{1+4\epsilon}\bar{t}}\frac{\epsilon}{\sqrt{1+4\epsilon}}[-e^{-\sqrt{1+4\epsilon}\bar{t}} \pm e^{\sqrt{1+4\epsilon}c_1}] = \frac{\epsilon}{\sqrt{1+4\epsilon}}[-1 \pm e^{\sqrt{1+4\epsilon}(\bar{t}+c_1)}]$$

Notice that $v(0) = \frac{1}{\alpha - \left[\frac{\sqrt{1+4\epsilon}-1}{2\epsilon}\right]}$

$$\frac{1}{\alpha - \left[\frac{\sqrt{1+4\epsilon}-1}{2\epsilon}\right]} = \frac{\epsilon}{\sqrt{1+4\epsilon}}[-1 \pm e^{\sqrt{1+4\epsilon}c_1}]$$

Thus

$$\left(\frac{\frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}} + 1}{\frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}} - 1}\right) = \pm e^{\sqrt{1+4\epsilon}c_1}$$

If the initial value $c_0 = \alpha c_{\text{in}} < C_{\text{eq}} = \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}-1}{2k\frac{V}{q}} \Leftrightarrow \frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}} - 1 < 0$

$$\left(\frac{\frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}} + 1}{\frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}} - 1}\right) = -e^{\sqrt{1+4\epsilon}c_1}, \quad c_1 = \frac{2\tanh^{-1}(\frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}})}{\sqrt{1+4\epsilon}} = \frac{2\tanh^{-1}(\frac{2kc_0(V/q)+1}{\sqrt{1+4kc_{\text{in}}(V/q)}})}{\sqrt{1+4kc_{\text{in}}(V/q)}}$$

$$\bar{C}_h(\bar{t}) = \frac{1}{v} = \frac{\sqrt{1+4\epsilon}}{2\epsilon}\left[\frac{2}{-1-e^{\sqrt{1+4\epsilon}(\bar{t}+c_1)}}\right]$$

$$\bar{C}(\bar{t}) = \bar{C}_h + \bar{C}_p = \frac{\sqrt{1+4\epsilon}}{2\epsilon}\left[\frac{2}{-1-e^{\sqrt{1+4\epsilon}(\bar{t}+c_1)}}\right] + \frac{\sqrt{1+4\epsilon}-1}{2\epsilon} = \frac{\sqrt{1+4\epsilon}\left(\frac{e^{\sqrt{1+4\epsilon}(\bar{t}+c_1)}-1}{e^{\sqrt{1+4\epsilon}(\bar{t}+c_1)}+1}\right) - 1}{2\epsilon}$$

$$= \frac{\sqrt{1+4\epsilon}\tanh\left(\frac{\sqrt{1+4\epsilon}}{2}(\bar{t}+c_1)\right) - 1}{2\epsilon}$$

If the initial value $c_0 = \alpha c_{\text{in}} > C_{\text{eq}} = \frac{\sqrt{1+4kc_{\text{in}}\frac{V}{q}}-1}{2k\frac{V}{q}} \Leftrightarrow \frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}} - 1 > 0$

$$\left(\frac{\frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}} + 1}{\frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}} - 1}\right) = +e^{\sqrt{1+4\epsilon}c_1}, \quad c_1 = \frac{2\coth^{-1}(\frac{2\epsilon\alpha+1}{\sqrt{1+4\epsilon}})}{\sqrt{1+4\epsilon}} = \frac{2\coth^{-1}(\frac{2kc_0(V/q)+1}{\sqrt{1+4kc_{\text{in}}(V/q)}})}{\sqrt{1+4kc_{\text{in}}(V/q)}}$$

$$\bar{C}_h(\bar{t}) = \frac{1}{v} = \frac{\sqrt{1+4\epsilon}}{2\epsilon}\left[\frac{2}{-1+e^{\sqrt{1+4\epsilon}(\bar{t}+c_1)}}\right]$$

$$\bar{C}(\bar{t}) = \bar{C}_h + \bar{C}_p = \frac{\sqrt{1+4\epsilon}}{2\epsilon}\left[\frac{2}{-1+e^{\sqrt{1+4\epsilon}(\bar{t}+c_1)}}\right] + \frac{\sqrt{1+4\epsilon}-1}{2\epsilon} = \frac{\sqrt{1+4\epsilon}\left(\frac{e^{\sqrt{1+4\epsilon}(\bar{t}+c_1)}+1}{e^{\sqrt{1+4\epsilon}(\bar{t}+c_1)}-1}\right) - 1}{2\epsilon}$$

$$= \frac{\sqrt{1+4\epsilon}\coth\left(\frac{\sqrt{1+4\epsilon}}{2}(\bar{t}+c_1)\right) - 1}{2\epsilon}$$

To sum up, and substitute with $\bar{C} \equiv C/c_{\text{in}}, \bar{t} \equiv t/(\frac{V}{q})$, we can obtain the same expression as (2)

Problem 7.

7. (Biogeography) The MacArthur-Wilson model of the dynamics of species (e.g., bird species) that inhabit an island located near a mainland was developed in the 1960s. Let N be the number of species in the source pool on the mainland, and let $S = S(t)$ be the number of species on the island. Assume that the rate of change of the number of species is

$$S' = \chi - \mu$$

where χ is colonization rate and μ is the extinction rate. In the MacArthur-Wilson model,

$$\chi = I\left(1 - \frac{S}{N}\right) \quad \text{and} \quad \mu = \frac{E}{N}S$$

where I and E are the maximum colonization and extinction rates, respectively.

(a) Over a long time, what is the expected equilibrium for the number of species inhabiting the island?
(b) Given $S(0) = S_0$, find an analytic formula for $S(t)$
(c) Suppose there are two islands, one large and one small, with the larger island having the smaller maximum extinction rate. Both have the same colonization rate. Show that the smaller island will eventually have fewer species.

solution
Establish the equation

$$\frac{dS}{dt} = \chi - \mu = I\left(1 - \frac{S}{N}\right) - \frac{E}{N}S = I - (\frac{I+E}{N})S$$

(a) Determine the expected equilibrium S_{eq} by setting $\frac{dS}{dt} = 0$

$$0 = I - \left(\frac{I+E}{N}\right)S_{\text{eq}}$$

The equilibrium S_{eq} is

$$S_{\text{eq}} = \left(\frac{I}{I+E}\right)N \tag{1}$$

(b) Given $S(0) = S_0$, find an analytic formula for $S(t)$

$$\frac{dS}{dt} + \left(\frac{I+E}{N}\right)S = I, \quad S(0) = S_0$$

Multiply the integrating factor $\mu(t) = e^{\int p(t)dt} = e^{\int (\frac{I+E}{N})dt} = e^{(\frac{I+E}{N})t}$

$$\frac{d[\mu(t)S]}{dt} = I\mu(t)$$

Integrate, then divide by $\mu(t)$

$$S(t) = \frac{1}{\mu(t)}\int I\mu(t)dt = e^{-(\frac{I+E}{N})t}I\int e^{(\frac{I+E}{N})t}dt = e^{-(\frac{I+E}{N})t}\left[\left(\frac{I}{I+E}\right)Ne^{(\frac{I+E}{N})t} + c_1\right]$$

Notice $S(0) = S_0$

$$S_0 = \left(\frac{I}{I+E}\right)N + c_1 \Rightarrow c_1 = S_0 - \left(\frac{I}{I+E}\right)N$$

In the end, $S(t)$ is

$$S(t) = \left(\frac{I}{I+E}\right) N + \left[S_0 - \left(\frac{I}{I+E}\right) N\right] e^{-(\frac{I+E}{N})t} \tag{2}$$

(c) Suppose there are two islands, one large and one small, with the larger island having the smaller maximum extinction rate. Both have the same colonization rate. Show that the smaller island will eventually have fewer species.
From (b), for any initial value, the final value $S(\infty)$ would always be

$$S(\infty) \equiv \lim_{t\to\infty} S(t) = \left(\frac{I}{I+E}\right) N$$

Now N is the number of species on the mainland, the same maximum colonization rate I
For the **large** island **1**, the maximum extinction rate E_1, the final species $S_1(\infty)$

$$S_1(\infty) = \left(\frac{I}{I+E_1}\right) N$$

For the **small** island **2**, the maximum extinction rate E_2, the final species $S_2(\infty)$

$$S_2(\infty) = \left(\frac{I}{I+E_2}\right) N$$

Notice the larger island having the smaller maximum extinction rate: $E_1 < E_2$

$$S_1(\infty) = \left(\frac{I}{I+E_1}\right) N > \left(\frac{I}{I+E_2}\right) N = S_2(\infty) \tag{3}$$

It shows that the **smaller** island **2** will eventually have **fewer** species $S_2(\infty)$

Bonus.

8. (Chemical reactors) A large industrial retention pond of volume V, initially free of pollutants, was subject to the inflow of a contaminant produced in the factory's processing plant. Over a period of b days the EPA found that the inflow concentration of the contaminant decreased linearly (in time) to zero from its initial initial value of a (grams per volume), its flow rate q (volume per day) being constant. During the b days the spillage rate to the local stream was also q. What is the concentration in the pond after b days? Take $V = 6000$ cubic meters, $b = 20$ days, $a = 0.03$ grams per cubic meter, and $q = 50$ cubic meters per day. With these data, how long would it take for the concentration in the pond to get below a required EPA level of 0.00001 grams per cubic meter if fresh water is pumped into the pond at the same flow rate, with the same spill over? [Use software to perform the calculations.]

solution

Establish equations, where c_{in}, c are the concentration of inflow, the concentration in the pond.

$$V\frac{dc}{dt} = c_{\text{in}}q - cq, \quad c(0) = 0, \quad c_{\text{in}} = \begin{cases} a - \dfrac{a}{b}t & 0 \le t \le b \\ 0 & t > b \end{cases}$$

(1) For the first stage ($0 \le t \le b$)

$$\frac{dc}{dt} + \frac{q}{V}c = \frac{q}{V}a(1 - \frac{t}{b}), \quad c(0) = 0, \quad (0 \le t \le b)$$

Multiply the integrating factor $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{q}{V}dt} = e^{\frac{q}{V}t}$, then integrate

$$\begin{aligned} c(t) &= \frac{1}{\mu(t)}\int_0^t \frac{q}{V}a(1 - \frac{s}{b})\mu(s)ds = ae^{-\frac{q}{V}t}\int_0^t \frac{q}{V}(1 - \frac{s}{b})e^{\frac{q}{V}s}ds \\ &= \left(\frac{e^{\frac{q}{V}t}(\frac{q}{V}b - \frac{q}{V}t + 1) - \frac{q}{V}b - 1}{\frac{q}{V}b}\right)ae^{-\frac{q}{V}t} \quad ,0 \le t \le b \end{aligned}$$

The concentration in the pond after b days, $c(b)$ is approximate **0.002239** (g/m^3)

$$c(b) = \left(\frac{e^{\frac{q}{V}b} - \frac{q}{V}b - 1}{\frac{q}{V}b}\right)ae^{-\frac{q}{V}b} = \left(\frac{e^{\frac{1}{6}} - \frac{1}{6} - 1}{\frac{1}{6}}\right)0.03e^{-\frac{1}{6}} \approx 0.002239(g/m^3) \tag{1}$$

(2) For the second stage ($b \le t$)

$$\frac{dc}{dt} + \frac{q}{V}c = 0, \quad c(b) = \left(\frac{e^{\frac{q}{V}b} - \frac{q}{V}b - 1}{\frac{q}{V}b}\right)ae^{-\frac{q}{V}b}, \quad (b \le t)$$

Similarly, solve the first order linear equation

$$c(t) = c(b)e^{-\frac{q}{V}(t-b)} = \left(\frac{e^{\frac{q}{V}b} - \frac{q}{V}b - 1}{\frac{q}{V}b}\right)ae^{-\frac{q}{V}t} \quad ,b \le t$$

Set $\epsilon = 0.00001$ grams per cubic meter, $c(t_\epsilon) = \epsilon$

$$t_\epsilon = \frac{\ln\left(\frac{e^{\frac{q}{V}b} - \frac{q}{V}b - 1}{\frac{q}{V}b}\right) + \ln(\frac{a}{\epsilon})}{\frac{q}{V}} = \frac{\ln\left(\frac{e^{\frac{1}{6}} - \frac{1}{6} - 1}{\frac{1}{6}}\right) + \ln(\frac{0.03}{0.00001})}{\frac{1}{120}} \approx 669.335(\text{day}) \tag{2}$$

It takes **670 days** for the concentration to get below a required EPA level of 0.00001 (g/m^3)

JOURNAL.

Write a list of types of differential equations that you have learned to solve in a differential equation course. For each type of differential equations, write the name of the method that solves this type of equations. (You do not need to describe the methods in detail, just give their names.)

solution
First order equations

(1) anti-derivatives $\frac{dx}{dt} = g(t)$

$$x(t) = \int_a^t g(s)ds + C$$

(2) separable equations $\frac{dx}{dt} = g(t)f(x)$

$$\int \frac{1}{f(x)}dx = \int g(t)dt + C$$

(3) linear equations $x' + p(t)x = q(t)$
Multiply the equation by the **integrating factor** $\mu(t) = e^{\int p(t)dt}$

$$\left(xe^{\int p(t)dt}\right)' = q(t)e^{\int p(t)dt}$$

Integrated to get

$$x(t) = e^{-\int p(t)dt}\left(\int q(t)e^{\int p(t)dt}dt + C\right)$$

(4) Bernoulli equations $x' + p(t)x = q(t)x^n$
The **substitution** $y = x^{1-n}$, multiply $(1-n)x^{-n}$

$$y' + (1-n)p(t)y = (1-n)q(t)$$

Reduce to a linear equation for $y = y(t)$
(5) Riccati equations $x' + p(t)x = q(t)x^2 + f(t)$
The general solution $x(t) = x_h(t) + x_p(t)$, where a particular solution $x_p(t)$ is known.
And the solution $x_h(t)$ for a Bernoulli equation (n=2)

$$x_h' + [p(t) - 2q(t)x_p]\, x_h = q(t)x_h^2$$

(6) homogeneous equations $x' = f(\frac{x}{t})$
The **substitution** $v \equiv \frac{x}{t}$

$$\frac{dv}{dx}v = f(v)$$

Reduce to a separable equation for $v = v(x)$
(7) exact equations $f(t,x) + g(t,x)x' = 0$, and $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial t}$
Then a potential function $H(x,t)$ exists, and $H(t,x) = c$

$$\frac{\partial H}{\partial t} = f(x,t), \quad \frac{\partial H}{\partial x} = g(x,t)$$

In the end, it gives

$$\begin{aligned} H(t,x) &= \int_{t_0}^t f(t',x_0)dt' + \int_{x_0}^x g(t,x')dx \\ &= \int_{t_0}^t f(t',x_0)dt' + \int_{x_0}^x \left[g(t_0,x') + \int_{t_0}^t \frac{\partial f(t',x')}{\partial x'}dt'\right] dx' = c \end{aligned}$$

Second order equations

(1) linear equations
- homogeneous equation, constant coefficients $x'' + a_1 x' + a_0 x = 0$
  The **characteristic equation**, roots are r_1, r_2
  $$r^2 + a_1 r + a_0 = 0$$
  If $r_1 = r_2$, the general solution $x(t) = (c_1 + c_2 t)e^{r_1 t}$
  Otherwise, the general solution $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
- homogeneous equation $x'' + a_1(t)x' + a_0(t)x = 0$, solution $x_1(t)$ known
  Wronskian $W(t) = W(t_0)e^{-\int_{t_0}^{t} a_1(\tau)d\tau}$, we can set $W(t_0)$ to be an any real number
  In the other way
  $$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = x_1 x_2' - x_1' x_2 = W(t)$$
  Reduce to a first order linear equation, where $p(t) = -\frac{x_1'}{x_1}$, $q(t) = \frac{W(t)}{x_1} = \frac{W(t_0)}{x_1}e^{-\int_{t_0}^{t} a_1(\tau)d\tau}$
  $$x_2' + p(t)x_2 = q(t)$$
  With the **substitution** $v(t) \equiv \frac{x_2(t)}{x_1}$, then integrate
  $$v(t) = \int \frac{dv}{dt}dt = \int \frac{W(t_0)}{x_1^2}e^{-\int_{t_0}^{t} a_1(\tau)d\tau}dt$$
- non-homogeneous equation $x'' + a_1(t)x' + a_0(t)x = f(t)$
  If we know the linear independent solutions $x_1(t), x_2(t)$ for $x'' + a_1(t)x' + a_0(t) = 0$
  The general solution $x(t) = x_h(t) + x_p(x)$, where $x_h(t) = c_1 x_1(t) + c_2 x_2(t)$
  variation of parameters: the particular solution $x_p(t) = c_1(t)x_1(t) + c_2(t)x_2(t)$ satisfies
  $$\begin{pmatrix} x_1 & x_2 \\ x_1' & x_2' \end{pmatrix}\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$
  Thus, use Cramer's rule and integrate, where $W(t) \equiv \left|\begin{smallmatrix} x_1 & x_2 \\ x_1' & x_2' \end{smallmatrix}\right| = W(t_0)e^{-\int_{t_0}^{t} a_1(\tau)d\tau}$
  $$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} -\int \frac{f(t)u_2(t)}{W(t)}dt \\ \int \frac{f(t)u_1(t)}{W(t)}dt \end{pmatrix}$$
- non-homogeneous, special function $f(t)$, constant coefficients $x'' + a_1 x' + a_0 x = f(t)$
  undetermined coefficients:
  $f(t)$ are α, $e^{\beta t}$, $\sin(\omega t)$, $\cos(\omega t)$, t^n, and sums, products of these common functions

Form of source function $f(t)$	Trial form of particular solution $x_p(t)$
α	A
$\alpha e^{\beta t}$	$Ae^{\beta t}$
Polynomial of degree n	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$
$\alpha \sin \omega t; \alpha \cos \omega t$	$A \sin \omega t + B \cos \omega t$
$\alpha e^{rt} \sin \omega t; \quad \alpha e^{rt} \cos \omega t$	$e^{rt}(A \sin \omega t + B \cos \omega t)$

(2) Cauchy-Euler equation $t^2 x'' + b_1 t x' + b_0 x = 0$
The **substitution** $p = \ln(t)$, it leads to
$$dx = x'dt = x'\frac{dt}{dp}dp = x'tdp$$
$$d^2x = x''dt^2 = d(dx) = d\left(x'tdp\right) = d\left(x't\right)dp = \left(\left[x''\frac{dt}{dp}dp\right]t + x'\frac{dt}{dp}dp\right)dp = \left[x''t^2 + x't\right]dp^2$$

Thus

$$\frac{dx}{dp} = tx', \quad \frac{d^2x}{dp^2} - \frac{dx}{dp} = t^2x''$$

Reduce to the homogeneous, constant coefficient equation

$$\frac{d^2x}{dp^2} + (b_1 - 1)\frac{dx}{dp} + b_0x = 0$$

The **indicial equation**

$$r(r-1) + b_1r + b_0 = r^2 + (b_1 - 1)r + b_0 = 0$$

(3) nonlinear equation

- special form $x'' = f(t, x')$
  The **substitution** $v \equiv x'$, it becomes first order equation

$$\frac{dv}{dt} = f(t, v)$$

- special form $x'' = f(x, x')$
  The **substitution** $v \equiv x'$, it leads to

$$x'' = \frac{dx'}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

  Reduce to the first order equation

$$v\frac{dv}{dx} = f(x, v)$$

Higher order equations

(1) linear equations

- homogeneous equation, constant coefficients $x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_0x = 0$
  The **characteristic equation**

$$r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0$$

  It has m roots $r_1, r_2, \cdots, r_m$, the multiplicity of which, respectively, is equal to $k_1, k_2, \cdots, k_m$

$$\begin{aligned} x(t) = {} & \left(c_1 + c_2t + \cdots + c_{k_1}t^{k_1-1}\right)e^{r_1t} + \cdots \\ & + \left(c_{n-k_m+1} + c_{n-k_m+2}t + \cdots + c_nt^{k_m-1}\right)e^{r_mt} \end{aligned}$$

- non-homogeneous equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = f(t)$
  Linear independent $x_1(t), \cdots, x_n(t)$ known for $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$
  The general solution $x(t) = x_h(t) + x_p(x)$, where $x_h(t) = \sum_{i=1}^n c_ix_i(t)$
  variation of parameters: the particular solution $x_p(t) = \sum_{i=1}^n c_i(t)x_i(t)$ satisfies

$$\begin{pmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t) & x_2^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{pmatrix} \begin{pmatrix} c_1'(t) \\ c_2'(t) \\ \vdots \\ c_n'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{pmatrix}$$

Thus, use Cramer's rule and integrate, where Wronskian $W(t) = W(t_0)e^{-\int_{t_0}^{t} a_{n-1}(\tau)d\tau}$, and $W_i(t)$ is the Wronskian determinant with the i-th column replaced by $\begin{bmatrix} 0 & 0 & \cdots & f(t) \end{bmatrix}^T$

$$\begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix} = \int \begin{pmatrix} \frac{W_1(t)}{W(t)} \\ \frac{W_2(t)}{W(t)} \\ \vdots \\ \frac{W_n(t)}{W(t)} \end{pmatrix} dt$$

- non-homogeneous, special $f(t)$, constant coefficients $x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_0 x = f(t)$ **undetermined coefficients**:
  $f(t)$ are α, $e^{\beta t}$, $\sin(\omega t)$, $\cos(\omega t)$, t^n, and sums, products of these common functions

Form of source function $f(t)$	Trial form of particular solution $x_p(t)$
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$\alpha e^{rt} \sin \omega t; \quad \alpha e^{rt} \cos \omega t$	$e^{rt}(A \sin \omega t + B \cos \omega t)$