Problem 5.

5. The dynamics of a nonlinear spring-mass system are described by

$$\begin{aligned} mx'' &= -ax' - kx^3 \\ x(0) &= 0, \quad mx'(0) = I \end{aligned}$$

where x is the displacement, $-ax'$ is a linear damping term, and $-kx^3$ is nonlinear restoring force. Initially, the displacement is zero and the mass m is given an impulse I that starts the motion.

(1) Determine the dimensions of the constants I, a, and k.
(2) Recast the problem into dimensionless form by selecting dimensionless variables $\tau = t/T, u = ax/I$, where the time scale T is yet to be determined.
(3) In the special case that the mass is very small, choose an appropriate time scale T and find the correct dimensionless model. (A small dimensionless parameter should occur on the terms involving the mass in the original model.)

solution

(1) the dimensions of I, a and k

$$[I] = MLT^{-1}, [a] = (MLT^{-2})/(LT^{-1}) = MT^{-1}, [k] = (MLT^{-2})/(L^3) = ML^{-2}T^{-2} \tag{1}$$

(2) Recast the problem with $\tau = t/T, u = ax/I$

$$\begin{aligned} &\tfrac{(I/a)}{T^2}(m\tfrac{d^2u}{d\tau^2}) = \tfrac{(I/a)}{T}(-a\tfrac{du}{d\tau}) + (I/a)^3(-ku^3) \\ &u(0) = 0, \quad \tfrac{(I/a)}{T}(m\tfrac{du}{d\tau}|_{\tau=0}) = I \end{aligned}$$

To simplify it

$$\begin{aligned} (\frac{m}{aT})\frac{d^2u}{d\tau^2} &= -\frac{du}{d\tau} - \frac{kTI^2}{a^3}u^3 \\ u(0) = 0, \quad \frac{du}{d\tau}|_{\tau=0} &= \frac{aT}{m} \end{aligned} \tag{2}$$

(3) When the mass m is small, the coefficient of the u'' term is small.
To make the restoring force term have coefficient 1.

$$\frac{kTI^2}{a^3} = 1 \Longrightarrow T = \frac{a^3}{kI^2}$$

Select $\epsilon \equiv \frac{m}{aT} = \frac{mkI^2}{a^4}$

$$\begin{aligned} \epsilon\frac{d^2u}{d\tau^2} &= -\frac{du}{d\tau} - u^3 \\ u(0) = 0, \quad \frac{du}{d\tau}|_{\tau=0} &= \frac{1}{\epsilon} \end{aligned} \tag{3}$$

Problem 9.

9. The temperature $T = T(t)$ of a chemical sample in a furnace at time t is governed by the initial value problem

$$\frac{dT}{dt} = qe^{-A/T} - k\left(T - T_f\right), \quad T(0) = T_0$$

where T_0 is the initial temperature of the sample, T_f is the temperature in the furnace, and q, k, and A are positive constants. The first term on the right side is the heat generation term, and the second is the heat loss term given by Newton's law of cooling.

(1) What are the dimensions of the constants q, k, A?
(2) Reduce the problem to dimensionless form using T_f as the temperature scale and choosing a time scale to be one appropriate to the case when the heat loss term is large compared to the heat generated by the reaction.

solution

(1) the dimensions of q, k, A

$$[q] = T^{-1}\Theta, [k] = T^{-1}, [A] = \Theta$$

(2) the dimensionless quantity π ($[\pi] = 1$) can be formed from q, k, A, T_f, T_0, t, T

$$[\pi] = [q^{p_1} k^{p_2} A^{p_3} T_f^{p_4} T_0^{p_5} t^{p_6} T^{p_7}] = 1 \tag{1}$$

Consider the fundamental dimension M, L, the dimension matrix A is

	q	k	A	T_f	T_0	t	T
T	-1	-1	0	0	0	1	0
Θ	1	0	1	1	1	0	1

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Do row operations S, the reduced matrix

$$SA = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

For equation $A\mathrm{p} = 0$, the number of independent solutions is $m - \mathrm{rank}(A) = 7 - 2 = 5$, where m is the number of dimensional quantities. The solution can be written in vector form as

$$\mathrm{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{pmatrix} = k_1 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_4 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k_5 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad k_1, k_2, k_3, k_4, k_5 \in \mathbb{R}$$

For $T(k_5 = 1)$, has no relationship with the coefficient q of heat generation item
$\Longrightarrow$ not include q and $t(k_4 = 0)$ in π_1: choose $(k_1, k_2, ..., k_5) = (0, -1, 0, 0, 1)$

$$\mathrm{p} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \pi_1 = \frac{1}{T_f}T, \quad T_c \equiv \frac{T}{\pi_1} = T_f$$

For $t(k_4 = 1)$, has no relationship with the coefficient q of heat generation item $\Longrightarrow$ not include q and $T(k_5 = 0)$ in π_2: choose $(k_1, k_2, ..., k_5) = (0, 0, 0, 1, 0)$

$$\mathrm{p} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \pi_2 = kt, \quad t_c \equiv \frac{t}{\pi_2} = \frac{1}{k}$$

Recast the problem with $\bar{T} = T/T_c = T/T_f, \tau = t/t_c = kt$

$$\frac{T_c}{t_c}(\frac{d\bar{T}}{d\tau}) = qe^{-\frac{1}{T_c}(A/\bar{T})} - T_c(k\left(\bar{T} - T_f/T_c\right)), \quad T_c\bar{T}(0) = T_0$$

To simplify it

$$\frac{d\bar{T}}{d\tau} = \frac{q}{kT_f}e^{-\frac{A}{T_f}(1/\bar{T})} - (\bar{T} - 1), \quad \bar{T}(0) = \frac{T_0}{T_f} \tag{2}$$

Set $\epsilon \equiv \frac{q}{kT_f}, E \equiv \frac{A}{T_f}, \alpha = \frac{T_0}{T_f}$

$$\frac{d\bar{T}}{d\tau} = \epsilon e^{-\frac{E}{T}} - (\bar{T} - 1), \quad \bar{T}(0) = \alpha \tag{3}$$

heat loss term >> the heat generated by the reaction, $\epsilon \approx 0$

$$\frac{d\bar{T}}{d\tau} = -(\bar{T} - 1), \quad \bar{T}(0) = \alpha \tag{4}$$

Problem 15.

15. The initial value problem for the damped pendulum equation is

$$\frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0$$
$$\theta(0) = \theta_0, \quad \theta'(0) = \omega_0$$

(1) Find three time scales and comment upon what process each involves.
(2) Non-dimensionalize the model with a time scale appropriate to expecting damping to have a small contribution.
(3) Non-dimensionalize the model with a time scale based on the fact that damping has an effect.

solution

(1) Find three time scales and comment upon what process each involves.
The dimensions of parameters and variables (note: angle θ, θ_0 have no dimension)

$$[k] = T^{-1}, [g] = LT^{-2}, [l] = L, [\omega_0] = T^{-1}, [t] = T$$

The dimension matrix

$$\begin{array}{c|ccccc} & k & g & l & \omega_0 & t \\ \hline L & 0 & 1 & 1 & 0 & 0 \\ T & -1 & -2 & 0 & -1 & 1 \end{array}, \quad A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ -1 & -2 & 0 & -1 & 1 \end{pmatrix}$$

Do row operations S, the reduced matrix is

$$SA = \begin{pmatrix} 1 & 0 & -2 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

For equation $A\mathrm{p} = 0$, the number of independent solutions is $m - \mathrm{rank}(A) = 5 - 2 = 3$, where m is the number of dimensional quantities. The solution can be written in vector form as

$$\mathrm{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = k_1 \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad k_1, k_2, k_3 \in \mathbb{R}$$

Select $(k_1, k_2, k_3) = (0, 0, 1), (-1/2, 0, 1), (0, 1, 1)$ respectively

$$\mathrm{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \pi = kt, \sqrt{\frac{g}{l}}t, \omega_0 t$$

The three time scales t_c

$$t_c \equiv \frac{t}{\pi} = \frac{1}{k}, \sqrt{\frac{l}{g}}, \frac{1}{\omega_0} \tag{1}$$

$\frac{1}{k}$: effect of the damping force
$\sqrt{\frac{l}{g}}$: effect of the gravity force
$\frac{1}{\omega_0}$: swing process, effect of initial radius velocity

(2) Expecting damping to have a small contribution.
Scale θ with $\theta_c = \theta_0$
Consider t_c, has no relationship with the coefficient k of damping item
$\Longrightarrow$ not include k in t_c, choose $t_c = \sqrt{\frac{l}{g}}$
Recast problem with $\bar{\theta} \equiv \theta/\theta_c, \bar{t} \equiv t/t_c$

$$\frac{\theta_c}{t_c^2}\left(\frac{d^2\bar{\theta}}{d\bar{t}^2}\right) + \frac{\theta_c}{t_c}\left(k\frac{d\bar{\theta}}{d\bar{t}}\right) + \left(\frac{g}{l}\sin(\theta_c\bar{\theta})\right) = 0$$
$$\theta_c\bar{\theta}(0) = \theta_0, \quad \frac{\theta_c}{t_c}\theta'(0) = \omega_0$$

To simplify it

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + k\sqrt{\frac{l}{g}}\frac{d\bar{\theta}}{d\bar{t}} + \frac{\sin(\theta_0\bar{\theta})}{\theta_0} = 0$$
$$\bar{\theta}(0) = 1, \quad \frac{d\bar{\theta}}{d\bar{t}}|_{\bar{t}=0} = \frac{\omega_0}{\theta_0}\sqrt{\frac{l}{g}} \tag{2}$$

Set $\epsilon \equiv k\sqrt{\frac{l}{g}}, \alpha \equiv \frac{\omega_0}{\theta_0}\sqrt{\frac{l}{g}}$

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \epsilon\frac{d\bar{\theta}}{d\bar{t}} + \frac{\sin(\theta_0\bar{\theta})}{\theta_0} = 0$$
$$\bar{\theta}(0) = 1, \quad \frac{d\bar{\theta}}{d\bar{t}}|_{\bar{t}=0} = \alpha \tag{3}$$

When $\epsilon \equiv k\sqrt{\frac{l}{g}} \to 0$

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{\sin(\theta_0\bar{\theta})}{\theta_0} = 0, \quad \bar{\theta}(0) = 1, \quad \frac{d\bar{\theta}}{d\bar{t}}|_{\bar{t}=0} = \alpha \tag{4}$$

(3) Based on the fact that damping has an effect.
Scale θ with $\theta_c = \theta_0$
Consider t_c, has no relationship with the coefficient $\frac{g}{l}$ of gravity item
$\Longrightarrow$ not include l, g in t_c, choose $t_c = \frac{1}{k}$
Recast problem with $\bar{\theta} \equiv \theta/\theta_c, \bar{t} \equiv t/t_c$, to simplify it

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{d\bar{\theta}}{d\bar{t}} + \frac{g}{lk^2}\frac{\sin(\theta_0\bar{\theta})}{\theta_0} = 0$$
$$\bar{\theta}(0) = 1, \quad \frac{d\bar{\theta}}{d\bar{t}}|_{\bar{t}=0} = \frac{\omega_0}{\theta_0 k} \tag{5}$$

Set $\epsilon \equiv \frac{g}{lk^2}, \alpha \equiv \frac{\omega_0}{\theta_0 k}$

$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \frac{d\bar{\theta}}{d\bar{t}} + \epsilon\frac{\sin(\theta_0\bar{\theta})}{\theta_0} = 0$$
$$\bar{\theta}(0) = 1, \quad \frac{d\bar{\theta}}{d\bar{t}}|_{\bar{t}=0} = \alpha \tag{6}$$

Bonus 10.

10. A ball of mass m is tossed upward with initial velocity V. Assuming the force caused by air resistance is proportional to the square of the velocity of the ball and the gravitational field is constant, formulate an initial value problem for the height of the ball at any time t. Choose characteristic length and time scales and recast the problem in dimensionless form.

solution
Set mass m, gravity g, resistance factor a, initial velocity V, time t, height h, from Newton's law

$$m\frac{d^2h}{dt^2} = -mg - a(\frac{dh}{dt})|\frac{dh}{dt}|, \quad h(0) = 0, \frac{dh}{dt}|_{t=0} = V \tag{1}$$

Dimensions of parameters and variables

$$[m] = M, [g] = LT^{-2}, [a] = ML^{-1}, [V] = LT^{-1}, [t] = T, [h] = L$$

Write the dimension matrix

$$\begin{array}{c|cccccc} & m & g & a & V & t & h \\ \hline M & 1 & 0 & 1 & 0 & 0 & 0 \\ L & 0 & 1 & -1 & 1 & 0 & 1 \\ T & 0 & -2 & 0 & -1 & 1 & 0 \end{array}, \quad A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & -2 & 0 & -1 & 1 & 0 \end{pmatrix}$$

Do row operations S, the reduced matrix

$$SA = \begin{pmatrix} 1 & 0 & 0 & 1/2 & 1/2 & 1 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 & -1 \end{pmatrix}$$

For equation $A\mathrm{p} = 0$, the number of independent solutions is $m - \mathrm{rank}(A) = 6 - 3 = 3$, where m is the number of dimensional quantities. The solution can be written in vector form as

$$\mathrm{p} = k_1 \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad k_1, k_2, k_3 \in \mathbb{R}$$

For $t(k_2 = 1)$, has no relationship with the coefficient a of resistance force item
$\Longrightarrow$ not include a and $h(k_3 = 0)$ in π_1: choose $(k_1, k_2, k_3) = (-1, 1, 0)$

$$\mathrm{p} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \pi_1 = \frac{g}{V}t, \quad t_c \equiv \frac{t}{\pi_1} = \frac{V}{g} \tag{2}$$

For $h(k_3 = 1)$, has no relationship with the coefficient a of resistance force item $\Longrightarrow$ not include a and $t(k_2 = 0)$ in π_2: choose $(k_1, k_2, k_3) = (-2, 0, 1)$

$$\mathrm{p} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \quad \pi_1 = \frac{g}{V^2}h, \quad h_c \equiv \frac{h}{\pi_2} = \frac{V^2}{g} \tag{3}$$

Recast (1) with $\bar{t} = t/t_c, \bar{h} = h/h_c$

$$\frac{h_c}{t_c^2}(m\frac{d^2\bar{h}}{d\bar{t}^2}) = -mg - \frac{h_c^2}{t_c^2}[a(\frac{d\bar{h}}{d\bar{t}})|\frac{d\bar{h}}{d\bar{t}}|], \quad \bar{h}(0) = 0, \frac{h_c}{t_c}(\frac{d\bar{h}}{d\bar{t}}|_{\bar{t}=0}) = V$$

To simplify it

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -1 - \frac{V^2 a}{mg}(\frac{d\bar{h}}{d\bar{t}})|\frac{d\bar{h}}{d\bar{t}}|, \quad \bar{h}(0) = 0, \frac{d\bar{h}}{d\bar{t}}|_{\bar{t}=0} = 1 \tag{4}$$

Set $\epsilon \equiv \frac{V^2 a}{mg}$

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -1 - \epsilon(\frac{d\bar{h}}{d\bar{t}})|\frac{d\bar{h}}{d\bar{t}}|, \quad \bar{h}(0) = 0, \frac{d\bar{h}}{d\bar{t}}|_{\bar{t}=0} = 1 \tag{5}$$

Journal.

(100-300 words, please type.) What is scaling for? What are characteristic scales? Write a paragraph that will help a MA 472 student to understand the purpose and method of scaling.

solution

Scaling is choosing right scales of quantities(length, time, etc).
It can helps when the quantities are too small or big. It also can sort out large terms and small terms. When establishing equations, it can tell us which one is important, which one can be neglected. It can make approximations by deleting small terms.

Characteristic scales is constructed by the system constant, to make the dimensionless quantities not too large or too small after Scaling.