Problem 1.

The speed v of a wave in deep water is determined by its wavelength λ and the acceleration g due to gravity. What does dimensional analysis imply regarding the relationship between v, λ, and g?

solution

It is fairly clear that the dimensions of them are

$$[v] = LT^{-1}, [\lambda] = L, [g] = LT^{-2}$$

From the Π theorem, if the dimensionless quantity π ($[\pi] = 1$) can be formed from v, λ, g

$$\pi = v^{p_1}\lambda^{p_2}g^{p_3} \tag{1}$$

$$[\pi] = [v^{p_1}\lambda^{p_2}g^{p_3}] = (LT^{-1})^{p_1}(L)^{p_2}(LT^{-2})^{p_3} = 1 \tag{2}$$

Consider the fundamental dimension L, T, the dimension matrix A is

$$\begin{array}{c|ccc} & v & \lambda & g \\ \hline L & 1 & 1 & 1 \\ T & -1 & 0 & -2 \end{array}, \quad A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \end{pmatrix}$$

The solution of $A\mathrm{p} = 0$ can be written in vector form as

$$\mathrm{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \mathrm{k}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \mathrm{k} \in \mathbb{R}$$

Here we select k=1, thus the physical law with the expression of π

$$F(\pi) = 0, \quad \pi = v^2\lambda^{-1}g^{-1} \tag{3}$$

It implies that, where $C = \sqrt{C'}$

$$\pi = \frac{v^2}{g\lambda} = C', \quad v = C\sqrt{g\lambda} \tag{4}$$

Problem 2.

An ecologist postulated that there is a relationship among the mass m, density ρ, volume V, and surface area S of certain animals. Discuss this conjecture in terms of dimensional analysis.

solution

It is fairly clear that the dimensions of them are

$$[m] = M, [\rho] = ML^{-3}, [V] = L^3, [S] = L^2$$

From the Π theorem, if the dimensionless quantity π ($[\pi] = 1$) can be formed from m, ρ, V, S

$$\pi = m^{p_1}\rho^{p_2}V^{p_3}S^{p_4} \tag{1}$$

$$[\pi] = [m^{p_1}\rho^{p_2}V^{p_3}S^{p_4}] = (M)^{p_1}(ML^{-3})^{p_2}(L^3)^{p_3}(L^2)^{p_4} = 1 \tag{2}$$

Consider the fundamental dimension M, L, the dimension matrix A is

	m	ρ	V	S
M	1	1	0	0
L	0	−3	3	2

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 2 \end{pmatrix}$$

For equation $A\mathrm{p} = 0$, the number of independent solutions is $m - \mathrm{rank}(A) = 4 - 2 = 2$, where m is the number of dimensional quantities. The solution can be written in vector form as

$$\mathrm{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \mathrm{k}_1 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \mathrm{k}_2 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 3 \end{pmatrix}, \quad \mathrm{k}_1, \mathrm{k}_2 \in \mathbb{R}$$

Here we select k_1, k_2=1, thus the physical law with the expressions of π_1, π_2

$$F(\pi_1, \pi_2) = 0, \quad \pi_1 = \frac{\rho V}{m}, \pi_2 = \frac{S^3}{V^2} \tag{3}$$

Solving for π_2 gives

$$\pi_2 = \frac{S^3}{V^2} = g(\pi_1) = g(\frac{\rho V}{m}) \tag{4}$$

Consequently, where $\hat{g}(\cdot) \equiv [g(\cdot)]^{\frac{1}{3}}$

$$S = V^{\frac{2}{3}}\hat{g}(\frac{\rho V}{m}) \tag{5}$$

Problem 4.

A physical phenomenon is described by the quantities P, l, m, t, and ρ, representing pressure, length, mass, time, and density, respectively. If there is a physical law

$$f(P, l, m, t, \rho) = 0$$

relating these quantities, show that there is an equivalent physical law of the form

$$G\left(l^3\rho/m, t^6P^3/m^2\rho\right) = 0$$

Find P in terms of an arbitrary function.

solution

It is fairly clear that the dimensions of them are

$$[P] = ML^{-1}T^{-2}, [l] = L, [m] = M, [t] = T, [\rho] = ML^{-3}$$

From the Π theorem, if the dimensionless quantity π ($[\pi] = 1$) can be formed from P, l, m, t, ρ

$$\pi = P^{p_1}l^{p_2}m^{p_3}t^{p_4}\rho^{p_5} \tag{1}$$

$$[\pi] = [P^{p_1}l^{p_2}m^{p_3}t^{p_4}\rho^{p_5}] = (ML^{-1}T^{-2})^{p_1}(L)^{p_2}(M)^{p_3}(T)^{p_4}(ML^{-3})^{p_5} = 1 \tag{2}$$

Consider the fundamental dimension M, L, T, the dimension matrix A is

	P	l	m	t	ρ
M	1	0	1	0	1
L	−1	1	0	0	−3
T	−2	0	0	1	0

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & -3 \\ -2 & 0 & 0 & 1 & 0 \end{pmatrix}$$

For equation $A\mathrm{p} = 0$, the number of independent solutions is $m - \text{rank}(A) = 5 - 3 = 2$, where m is the number of dimensional quantities. The solution can be written in vector form as

$$\mathrm{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = \mathrm{k}_1 \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \mathrm{k}_2 \begin{pmatrix} 3 \\ 0 \\ -2 \\ 6 \\ -1 \end{pmatrix}, \quad \mathrm{k}_1, \mathrm{k}_2 \in \mathbb{R}$$

Here we select k_1, k_2=1, thus the physical law with the expressions of π_1, π_2

$$G(\pi_1, \pi_2) = 0, \quad \pi_1 = \frac{l^3\rho}{m}, \pi_2 = \frac{t^6P^3}{m^2\rho} \tag{3}$$

Solving for π_2 gives

$$\pi_2 = \frac{t^6P^3}{m^2\rho} = g(\pi_1) = g(\frac{l^3\rho}{m}) \tag{4}$$

Consequently, where $\hat{g}(\cdot) \equiv [g(\cdot)]^{\frac{1}{3}}$

$$P = \frac{m^{\frac{2}{3}}\rho^{\frac{1}{3}}}{t^2}\hat{g}(\frac{l^3\rho}{m}) \tag{5}$$

Problem 9.

In modeling the digestion process in insects, it is believed that digestion yield rate Y, in mass per time, is related to the concentration C of the limiting nutrient, the residence time T in the gut, the gut volume V, and the rate of nutrient breakdown r, given in mass per time per volume. Show that for fixed T, r, C, the yield is positively related to the gut volume.

solution

It is fairly clear that the dimensions of three of them are

$$[T] = T, [r] = ML^{-3}T^{-1}, [C] = ML^{-3}, [V] = L^3, [Y] = MT^{-1}$$

From the Π theorem, if the dimensionless quantity π ($[\pi] = 1$) can be formed from T, r, C, V, Y

$$\pi = T^{p_1} r^{p_2} C^{p_3} V^{p_4} Y^{p_5} \tag{1}$$

$$[\pi] = [T^{p_1} r^{p_2} C^{p_3} V^{p_4} Y^{p_5}] = (T)^{p_1}(ML^{-3}T^{-1})^{p_2}(ML^{-3})^{p_3}(L^3)^{p_4}(MT^{-1})^{p_5} = 1 \tag{2}$$

Consider the fundamental dimension M, L, T, the dimension matrix A is

	T	r	C	V	Y
M	0	1	1	0	1
L	0	−3	−3	3	0
T	1	−1	0	0	−1

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & -3 & -3 & 3 & 0 \\ 1 & -1 & 0 & 0 & -1 \end{pmatrix}$$

For equation $A\mathrm{p} = 0$, the number of independent solutions is $m - \mathrm{rank}(A) = 5 - 3 = 2$, where m is the number of dimensional quantities. The solution can be written in vector form as

$$\mathrm{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = \mathrm{k}_1 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mathrm{k}_2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathrm{k}_1, \mathrm{k}_2 \in \mathbb{R}$$

Here we select k_1, k_2=1, thus the physical law with the expressions of π_1, π_2

$$F(\pi_1, \pi_2) = 0, \quad \pi_1 = \frac{C}{Tr}, \pi_2 = \frac{TY}{CV} \tag{3}$$

Solving for π_2 gives

$$\pi_2 = \frac{TY}{CV} = g(\pi_1) = g(\frac{C}{Tr}) \tag{4}$$

Consequently

$$Y = \frac{CV}{T} g(\frac{C}{Tr}) = \Big[\frac{C}{T} g(\frac{C}{Tr})\Big] V \tag{5}$$

For fixed T, r, C, the coefficient $\Big[\frac{C}{T} g(\frac{C}{Tr})\Big]$ is a constant value

$$Y = \mathrm{const} \cdot V \tag{6}$$

It shows that the yield Y is positively related to the gut volume.

PROBLEM 13.

Did you ever wonder how fast a long line of dominos topple over?
Let us imagine an experiment where we set up a line of dominos with spacing d between them. Further, we assume a typical domino has height h and thickness τ. We seek a formula that relates these quantities, the gravitational constant g, and the velocity v.

(1) Use dimensional analysis to show that

$$v = \sqrt{gh}F\left(\frac{d}{h}, \frac{\tau}{h}\right)$$

Assume τ/h is very small and can be neglected. What does the law become?

(2) Experiments have been performed that show the graph of $v/\sqrt{gh}$ vs. d/h is approximately constant, 1.5, for d/h varying over the range 0 to 0.8.
Using $h = 0.05$ meters, what is the velocity of the toppling dominos?

solution

It is fairly clear that the dimensions of them are

$$[d] = L, [h] = L, [\tau] = L, [g] = LT^{-2}, [v] = LT^{-1}$$

From the Π theorem, if the dimensionless quantity π ($[\pi] = 1$) can be formed from d, h, τ, g, v

$$\pi = d^{p_1} h^{p_2} \tau^{p_3} g^{p_4} v^{p_5} \tag{1}$$

$$[\pi] = [d^{p_1} h^{p_2} \tau^{p_3} g^{p_4} v^{p_5}] = (L)^{p_1}(L)^{p_2}(L)^{p_3}(LT^{-2})^{p_4}(LT^{-1})^{p_5} = 1 \tag{2}$$

Consider the fundamental dimension L, T, the dimension matrix A is

	d	h	τ	g	v
L	1	1	1	1	1
T	0	0	0	−2	−1

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & -1 \end{pmatrix}$$

For equation $A\mathrm{p} = 0$, the number of independent solutions is $m - \mathrm{rank}(A) = 5 - 2 = 3$, where m is the number of dimensional quantities. The solution can be written in vector form as

$$\mathrm{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = \mathrm{k}_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathrm{k}_2 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mathrm{k}_3 \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \quad \mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3 \in \mathbb{R}$$

Here we select k_1, k_2, k_3=1, thus the physical law with the expressions of π_1, π_2

$$G(\pi_1, \pi_2, \pi_3) = 0, \quad \pi_1 = \frac{d}{h}, \pi_2 = \frac{\tau}{h}, \pi_3 = \frac{v^2}{gh} \tag{3}$$

Solving for π_3 gives

$$\pi_3 = \frac{v^2}{gh} = g(\pi_1, \pi_2) = g\left(\frac{d}{h}, \frac{\tau}{h}\right) \tag{4}$$

Consequently, where $F(\cdot) \equiv [g(\cdot)]^{\frac{1}{2}}$

$$v = \sqrt{gh}F\left(\frac{d}{h}, \frac{\tau}{h}\right) \tag{5}$$

Assume τ/h is very small and can be neglected, $F\left(\frac{d}{h}, \frac{\tau}{h}\right)$ can be replaced by $f\left(\frac{d}{h}\right)$

$$v = \sqrt{gh} f\left(\frac{d}{h}\right) \tag{6}$$

$v/\sqrt{gh}$ is approximately constant, 1.5, for d/h varying over the range 0 to 0.8.

$$\frac{v}{\sqrt{gh}} = f\left(\frac{d}{h}\right) \approx 1.5 \tag{7}$$

Eventually, here we use $g \approx 9.8(m/s^2), h = 0.05(m)$, the velocity is

$$v \approx \sqrt{gh} \times 1.5 \approx 0.7 \times 1.5 = 1.05(m/s) \tag{8}$$

Bonus.

Tsunamis are often caused by underwater earthquakes or landslides far away from the shore and travel fast across the ocean. It has been observed that the deeper the ocean the faster the water wave travels. However, as can be easily seen, there is no law relating only the speed v of the wave and the depth of the ocean d.

(1) What other quantity or quantities should be included? Why? Derive a law with these additional quantities.
(2) The massive tsunami caused by an earthquake in Japan on March 11, 2011 traveled in the Pacific Ocean at an average speed of 837 kilometers/hour. Knowing that the the average depth of the Pacific Ocean is 4,000 meters, how fast will a tsunami travel in the Atlantic Ocean whose average depth is 3,300 meters?

solution
The gravity acceleration g and the depth d should be included.
Reason: v is related to the kinetic energy, which has strong correlation with potential energy, that is a function of d and g.
It is fairly clear that the dimensions of them are

$$[v] = LT^{-1}, [d] = L, [g] = LT^{-2}$$

From the Π theorem, if the dimensionless quantity π ($[\pi] = 1$) can be formed from v, d, g

$$\pi = v^{p_1} d^{p_2} g^{p_3} \tag{1}$$

$$[\pi] = [v^{p_1} d^{p_2} g^{p_3}] = (LT^{-1})^{p_1} (L)^{p_2} (LT^{-2})^{p_3} = 1 \tag{2}$$

Consider the fundamental dimension L, T, the dimension matrix A is

	v	d	g
L	1	1	1
T	-1	0	-2

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \end{pmatrix}$$

The solution of $A\mathrm{p} = 0$ can be written in vector form as

$$\mathrm{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \mathrm{k} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \mathrm{k} \in \mathbb{R}$$

Here we select k=1, thus the physical law with the expression of π

$$F(\pi) = 0, \quad \pi = \frac{v^2}{gd} \tag{3}$$

It implies that, where $C = \sqrt{C'}$

$$\pi = \frac{v^2}{gd} = C', \quad v = C\sqrt{gd} \tag{4}$$

So, the tsunamis speed v_2 where the depth $d_2 = 3300(m)$ should be, when speed $v_1 = 837(km/h)$ where the depth $d_1 = 4000(m)$

$$\frac{v_2}{v_1} = \frac{C\sqrt{gd_2}}{C\sqrt{gd_1}} = \sqrt{\frac{d_2}{d_1}} \Longrightarrow v_2 = v_1\sqrt{\frac{d_2}{d_1}} = 837 \times \sqrt{\frac{3300}{4000}} \approx 760(km/h) \tag{5}$$

Journal.

(100-300 words.) What is the big idea behind the dimensional analysis? What can the dimensional analysis achieve? When is it effective?

Big idea behind it

- Quantities have the same dimension can be compared to other quantities of the same kind.
- Any physically meaningful equation have the same dimensions on its left and right sides.

What it achieves

- Serving as a plausibility check on derived equations and computations.
- Serving as a guide and constraint in deriving equations that may describe a physical system.
- Determining the nature of physical phenomena when the equations were not known.

When it is effective

- When in the absence of a more rigorous derivation.
- When the physical equations were not known.
- When checking on derived equations and computations.