Bonus.

16. (Circuits) An RCL circuit with a nonlinear resistance, where the voltage drop across the resistor is a nonlinear function of current, can be modeled by the Van der Pol equation

$$x'' + \rho\left(x^2 - 1\right)x' + x = 0$$

where ρ is a positive constant, and $x(t)$ is the current.

a) In the phase plane, show that the origin is an unstable equilibrium.
b) Sketch the nullclines and the vector field. What are the possible dynamics? Is there a limit cycle?

(Problem 16ab on Page 111, PDF Page 134)

solution
With the substitution $y = x'$

$$x' = y$$
$$y' = x'' = -x - \rho(x^2 - 1)y$$

The nullclines: $0 = y, 0 = -x - \rho(x^2 - 1)y$
the only critical point is $(0, 0)$

For the direction in the separated regions:
region 1: $y > 0, -x - \rho(x^2 - 1)y > 0$
$(x', y') = (y, -x - \rho(x^2 - 1)y) = (+, +)$
region 2: $y < 0, -x - \rho(x^2 - 1)y > 0$
$(x', y') = (y, -x - \rho(x^2 - 1)y) = (-, +)$
region 3: $y < 0, -x - \rho(x^2 - 1)y < 0$
$(x', y') = (y, -x - \rho(x^2 - 1)y) = (-, -)$
region 4: $y > 0, -x - \rho(x^2 - 1)y < 0$
$(x', y') = (y, -x - \rho(x^2 - 1)y) = (+, -)$

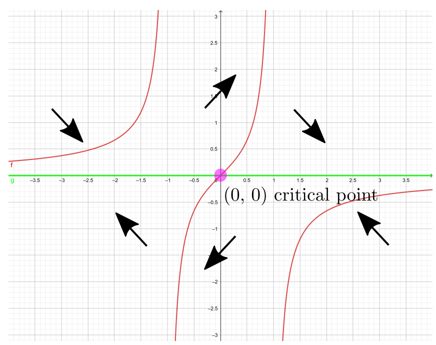

Figure 6. phase diagram $\rho = 1$

Consider the Jacobian J

$$J = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 - 2\rho xy & -\rho(x^2 - 1) \end{pmatrix}$$

a) At the critical point $(0, 0)$,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \rho \end{pmatrix}, \quad \det(J) \neq 0, \quad |J - \lambda I| = \lambda^2 - \rho\lambda + 1 = \lambda^2 - p\lambda + q = 0$$

With **Vieta theorem**, $\Delta = \rho^2 - 4, p = \lambda_1 + \lambda_2 = \text{Re}(\lambda_1) + \text{Re}(\lambda_2) = \rho > 0, q = 1$
if stable $\Rightarrow \text{Re}(\lambda_1) \leq 0, \text{Re}(\lambda_2) \leq 0 \Rightarrow \text{Re}(\lambda_1) + \text{Re}(\lambda_2) \leq 0$, there is a conflict, (0, 0) is **unstable**

b) The the nullclines and the vector field are displayed in the figure above

(1) $\Delta = \rho^2 - 4 = 0 \Rightarrow \rho = 2$
repeated eigenvalues $\lambda_1 = \lambda_2 = 1$, solution $c_1 w e^{\lambda_1 t} + c_2(w + vt)e^{\lambda_1 t}$
the critical point $(0, 0)$ is an **unstable improper node**

(2) $\Delta = \rho^2 - 4 > 0 \Rightarrow 2 < \rho$
real eigenvalues $\lambda_1 > \lambda_2 > 0$
the critical point $(0,0)$ is an **unstable node**

(3) $\Delta = \rho^2 - 4 < 0 \Rightarrow 0 < \rho < 2$
complex eigenvalues $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = \frac{p}{2} = \frac{\rho}{2} > 0$
the critical point $(0,0)$ is an **unstable spiral**

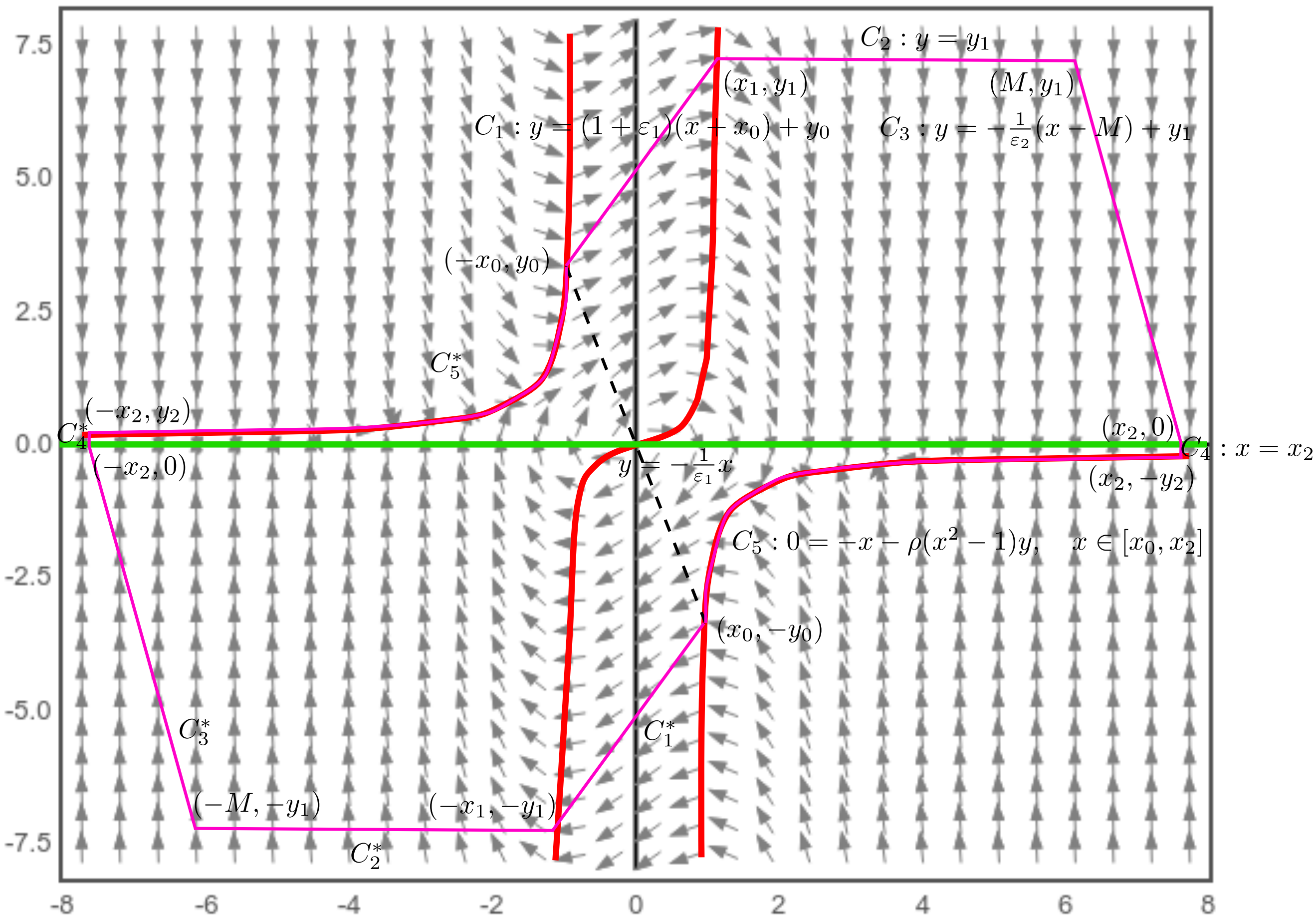

FIGURE 7. the simple closed curve $\boldsymbol{C}$, where $(x', y')^T \cdot \vec{n} < 0$

(Note: **Green**: x nullcline; **Red**: y nullcline; **Magenta**: the simple closed curve $\boldsymbol{C}$)

Theorem (Poincare-Bendixson Ring Domain Theorem). *Suppose $\boldsymbol{R}$ is the finite region of the plane lying between two simple closed curves $\boldsymbol{C}$ and $\bar{\boldsymbol{C}}$, and $\boldsymbol{F}$ is the velocity vector field for the system $x' = f(x,y) \quad y' = g(x,y)$. If*
(i) at each point of $\boldsymbol{C}$ and $\bar{\boldsymbol{C}}$, the field $\boldsymbol{F}$ points toward the interior of $\boldsymbol{R}$, and
(ii) $\boldsymbol{R}$ contains no critical points,
*then the system has a **closed trajectory** lying inside $\boldsymbol{R}$*

See the MIT limit cycle note or go to the url: `https://math.mit.edu/~jorloff/suppnotes/suppnotes03/lc.pdf`

For **Poincare-Bendixson Ring Domain Theorem**, we construct the simple closed curves first.
curve $\boldsymbol{C}$ consists of C_1, C_2, C_3, C_4, C_5 and the symmetrical curves $C_1^*, C_2^*, C_3^*, C_4^*, C_5^*$ of (0, 0)
For (x, y) on $\boldsymbol{C}$, always holds $(x', y')^T \cdot \vec{n} < 0$, here (x', y'): velocity field, $\vec{n}$: normal vector at (x, y)
Namely, for all (x, y) on $\boldsymbol{C}$, velocity field field (x', y') points toward the inside of $\boldsymbol{C}$
Short explanation for $(x', y')^T \cdot \vec{n} < 0$ on C_1, C_2, C_3, C_4, C_5 as follows:

(1) $C_1 : y = (\rho + \varepsilon_1)(x + x_0) + y_0$
where $(-x_0, y_0)$ is the intersection of $y = -\frac{1}{\varepsilon_1}x$ and left branch of $0 = -x - \rho(x^2 - 1)y$
ε_1 must satisfy, at $(-x_0, y_0)$: $\frac{dy}{dx}|_{(-x_0,y_0)}$ slope of $0 = -x - \rho(x^2 - 1)y > (\rho + \varepsilon_1)$ slope of C_1

$$(-x_0, y_0) = \left(-\sqrt{1 + \frac{\varepsilon_1}{\rho}}, \sqrt{1 + \frac{\varepsilon_1}{\rho}}/\varepsilon_1\right), \quad \text{left brach } 0 = -x-\rho(x^2-1)y \Leftrightarrow y = -\frac{1}{2\rho}\left(\frac{1}{x-1} + \frac{1}{x+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{\rho}\frac{x^2+1}{(x^2-1)^2}, \quad \frac{dy}{dx}|_{(-x_0,y_0)} = \frac{1}{\rho}\frac{2 + \frac{\varepsilon_1}{\rho}}{(\frac{\varepsilon_1}{\rho})^2} = \frac{2\rho + \varepsilon_1}{\varepsilon_1^2} > (\rho + \varepsilon_1) \Leftarrow \rho > 0 > -\varepsilon_1\left(\frac{1 - \varepsilon_1^2}{2 - \varepsilon_1^2}\right)$$

We can find $0 < \varepsilon_1 < 1$ to satisfy it, for $\forall \rho > 0$, it always holds

$$(x', y')^T = \left(y, -x - \rho(x^2 - 1)y\right)^T, \quad \vec{n} = (-\rho - \varepsilon_1, 1)^T$$
$$(x', y')^T \cdot \vec{n} = -\left[(\varepsilon_1 y + x) + \rho x^2\right] < 0$$

(2) $C_2 : y = y_1$
where (x_1, y_1) is the intersection of C_1 and center branch of $0 = -x - \rho(x^2 - 1)y$
for (x, y) on C_2, it has $x > x_1 > 0, y = y_1 > 0$, always holds

$$(x', y')^T = \left(y, -x - \rho(x^2 - 1)y\right)^T, \quad \vec{n} = (0, 1)^T$$

$$(x', y')^T \cdot \vec{n} = -x-\rho(x^2-1)y = \left[-x_1 - \rho(x_1^2 - 1)y_1\right] - (x-x_1) - (x^2-x_1^2)y_1 = -(x-x_1) - (x^2-x_1^2)y_1 < 0$$

(3) $C_3 : y = -\frac{1}{\varepsilon_2}(x - M) + y_1$
where (M, y_1) satisfies $M > \sqrt{1 + \frac{1}{\rho\varepsilon_2}}$
for (x, y) on C_3, it has $x > M, y > 0$, always holds

$$(x', y')^T = \left(y, -x - \rho(x^2 - 1)y\right)^T, \quad \vec{n} = (1, \varepsilon_2)^T$$

$$(x', y')^T \cdot \vec{n} = -\varepsilon_2 x - y\left[\rho\varepsilon_2(x^2 - 1) - 1\right] = -\varepsilon_2 x - y\left[\rho\varepsilon_2(M^2 - 1) - 1\right] - y\rho\varepsilon_2(x^2 - M^2) < 0$$

(4) $C_4 : x = x_2$
where $(x_2, 0)$ is the intersection of C_3 and $y = 0$
for (x, y) on C_4, it has $x = x_2, y < 0$, always holds

$$(x', y')^T = \left(y, -x - \rho(x^2 - 1)y\right)^T, \quad \vec{n} = (1, 0)^T$$
$$(x', y')^T \cdot \vec{n} = y < 0$$

(5) $C_5 : 0 = -x - \rho(x^2 - 1)y$ right branch
where $(x_2, -y_2)$ is the intersection of C_4 and $0 = -x - \rho(x^2 - 1)y$ right branch
for (x, y) on C_5, it has $x > x_0 = \sqrt{1 + \frac{\varepsilon_1}{\rho}} > 1, y < 0$, always holds

$$y = -\frac{1}{2\rho}\left(\frac{1}{x-1} + \frac{1}{x+1}\right), \quad \frac{dy}{dx} = \frac{1}{\rho}\frac{x^2+1}{(x^2-1)^2}$$

$$(x', y')^T = \left(y, -x - \rho(x^2 - 1)y\right)^T = (y, 0)^T, \quad \vec{n} = (\frac{dy}{dx}, -1)^T$$

$$(x', y')^T \cdot \vec{n} = -\left[\frac{1}{\rho}\frac{x^2+1}{(x^2-1)^2}\right](-y) < 0$$

To sum up, $(x', y')^T \cdot \vec{n} < 0$ holds on C_1, C_2, C_3, C_4, C_5, the velocity field is symmetrical to (0, 0). For the symmetrical curves $C_1^*, C_2^*, C_3^*, C_4^*, C_5^*$ of (0, 0), $(x', y')^T \cdot \vec{n} < 0$ holds on $C_1^*, C_2^*, C_3^*, C_4^*, C_5^*$. Thus, $(x', y')^T \cdot \vec{n} < 0$ holds for almost all points on entire $\boldsymbol{C}$
At each point of $\boldsymbol{C}$, the field $\boldsymbol{F} = (x', y')^T$ points toward the inside of $\boldsymbol{C}$
Now, consider to construct the simple closed curve $\bar{\boldsymbol{C}}$ (**Magenta**) near the critical point (0, 0)

(1) $\rho > 2, \Delta < 0$, construct $\bar{\boldsymbol{C}}$ as below
eigenvalues $\lambda_{1,2} = \frac{p \pm \sqrt{p^2-4}}{2} > 0$, with $(J - \lambda_{1,2})w_{1,2} = 0$, find $w_{1,2} = [1, \frac{p \pm \sqrt{p^2-4}}{2}]^T$
then $X(t) = (x, y)^T = c_1 e^{\lambda_1 t} w_1 + c_2 e^{\lambda_2 t} w_2$, $(x', y')^T = c_1 \lambda_1 e^{\lambda_1 t} w_1 + c_2 \lambda_2 e^{\lambda_2 t} w_2$
notice that coefficient $(c_1 \lambda_1 e^{\lambda_1 t}, c_2 \lambda_2 e^{\lambda_2 t})$ has the same signs as $(c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}), (c_1, c_2)$
the normal vector at (x, y) is $\vec{n} = \operatorname{sgn}(c_1)\frac{w_1}{|w_1|} + \operatorname{sgn}(c_2)\frac{w_2}{|w_2|}$, it always holds

$$\begin{aligned}
(x', y')^T \cdot \vec{n} &= \left(c_1 \lambda_1 e^{\lambda_1 t} w_1 + c_2 \lambda_2 e^{\lambda_2 t} w_2\right) \cdot \left(\operatorname{sgn}(c_1)\frac{w_1}{|w_1|} + \operatorname{sgn}(c_2)\frac{w_2}{|w_2|}\right) \\
&= \left(c_1 \lambda_1 e^{\lambda_1 t}|w_1|\frac{w_1}{|w_1|} + c_2 \lambda_2 e^{\lambda_2 t}|w_2|\frac{w_2}{|w_2|}\right) \cdot \left(\operatorname{sgn}(c_1 \lambda_1 e^{\lambda_1 t}|w_1|)\frac{w_1}{|w_1|} + \operatorname{sgn}(c_2 \lambda_2 e^{\lambda_2 t}|w_2|)\frac{w_2}{|w_2|}\right) \\
&= (k_1 \vec{e}_1 + k_2 \vec{e}_2) \cdot (\operatorname{sgn}(k_1)\vec{e}_1 + \operatorname{sgn}(k_2)\vec{e}_2) \quad (\text{where } k_1 \equiv c_1 \lambda_1 e^{\lambda_1 t}|w_1|, \vec{e}_1 \equiv \frac{w_1}{|w_1|}) \\
&= |k_1| + |k_2| + (|k_1| + |k_2|) \operatorname{sgn}(k_1 k_2)\vec{e}_1 \cdot \vec{e}_2 = (|k_1| + |k_2|)\left(1 + \operatorname{sgn}(k_1 k_2)\vec{e}_1 \cdot \vec{e}_2\right) > 0
\end{aligned}$$

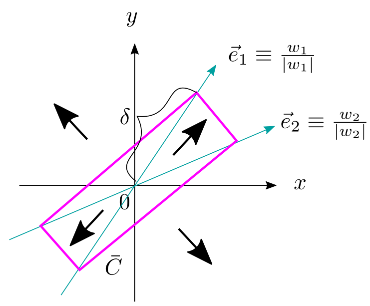

FIGURE 8. the simple closed curve $\bar{\boldsymbol{C}}$ (**Magenta**) near (0,0) for $\rho > 2$

(2) $0 < \rho \le 2, \Delta \le 0$, construct $\bar{\boldsymbol{C}}$ as below
it always holds $(x', y')^T \cdot \vec{n} > 0$

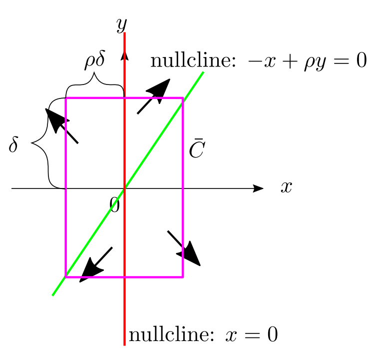

FIGURE 9. the simple closed curve $\bar{\boldsymbol{C}}$ (**Magenta**) near (0,0) for $0 < \rho \le 2$

At each point of $\bar{\boldsymbol{C}}$, the field $\boldsymbol{F} = (x', y')^T$ points toward the outside of $\bar{\boldsymbol{C}}$
Conclusion: with **Poincare-Bendixson Ring Domain Theorem**,
there is a **closed trajectory** (**limit cycle**) inside $\boldsymbol{R}$ between two simple closed curves $\boldsymbol{C}$ and $\bar{\boldsymbol{C}}$