JOURNAL.

Write a list of types of differential equations that you have learned to solve in a differential equation course. For each type of differential equations, write the name of the method that solves this type of equations. (You do not need to describe the methods in detail, just give their names.)

solution

First order equations

(1) anti-derivatives $\frac{dx}{dt} = g(t)$

$$x(t) = \int_a^t g(s)ds + C$$

(2) separable equations $\frac{dx}{dt} = g(t)f(x)$

$$\int \frac{1}{f(x)}dx = \int g(t)dt + C$$

(3) linear equations $x' + p(t)x = q(t)$

Multiply the equation by the **integrating factor** $\mu(t) = e^{\int p(t)dt}$

$$\left(xe^{\int p(t)dt}\right)' = q(t)e^{\int p(t)dt}$$

Integrated to get

$$x(t) = e^{-\int p(t)dt}\left(\int q(t)e^{\int p(t)dt}dt + C\right)$$

(4) Bernoulli equations $x' + p(t)x = q(t)x^n$

The **substitution** $y = x^{1-n}$, multiply $(1-n)x^{-n}$

$$y' + (1-n)p(t)y = (1-n)q(t)$$

Reduce to a linear equation for $y = y(t)$

(5) Riccati equations $x' + p(t)x = q(t)x^2 + f(t)$

The general solution $x(t) = x_h(t) + x_p(t)$, where a particular solution $x_p(t)$ is known.

And the solution $x_h(t)$ for a Bernoulli equation (n=2)

$$x_h' + \left[p(t) - 2q(t)x_p\right]x_h = q(t)x_h^2$$

(6) homogeneous equations $x' = f(\frac{x}{t})$

The **substitution** $v \equiv \frac{x}{t}$

$$\frac{dv}{dx}v = f(v)$$

Reduce to a separable equation for $v = v(x)$

(7) exact equations $f(t,x) + g(t,x)x' = 0$, and $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial t}$

Then a potential function $H(x,t)$ exists, and $H(t,x) = c$

$$\frac{\partial H}{\partial t} = f(x,t), \quad \frac{\partial H}{\partial x} = g(x,t)$$

In the end, it gives

$$\begin{aligned} H(t,x) &= \int_{t_0}^t f(t',x_0)dt' + \int_{x_0}^x g(t,x')dx \\ &= \int_{t_0}^t f(t',x_0)dt' + \int_{x_0}^x \left[g(t_0,x') + \int_{t_0}^t \frac{\partial f(t',x')}{\partial x'}dt'\right]dx' = c \end{aligned}$$

Second order equations

(1) linear equations
- homogeneous equation, constant coefficients $x'' + a_1 x' + a_0 x = 0$
  The **characteristic equation**, roots are r_1, r_2
  $$r^2 + a_1 r + a_0 = 0$$
  If $r_1 = r_2$, the general solution $x(t) = (c_1 + c_2 t)e^{r_1 t}$
  Otherwise, the general solution $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
- homogeneous equation $x'' + a_1(t)x' + a_0(t)x = 0$, solution $x_1(t)$ known
  Wronskian $W(t) = W(t_0)e^{-\int_{t_0}^{t} a_1(\tau)d\tau}$, we can set $W(t_0)$ to be an any real number
  In the other way
  $$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = x_1 x_2' - x_1' x_2 = W(t)$$
  Reduce to a first order linear equation, where $p(t) = -\frac{x_1'}{x_1}$, $q(t) = \frac{W(t)}{x_1} = \frac{W(t_0)}{x_1}e^{-\int_{t_0}^{t} a_1(\tau)d\tau}$
  $$x_2' + p(t)x_2 = q(t)$$
  With the **substitution** $v(t) \equiv \frac{x_2(t)}{x_1}$, then integrate
  $$v(t) = \int \frac{dv}{dt}dt = \int \frac{W(t_0)}{x_1^2}e^{-\int_{t_0}^{t} a_1(\tau)d\tau}dt$$
- non-homogeneous equation $x'' + a_1(t)x' + a_0(t)x = f(t)$
  If we know the linear independent solutions $x_1(t), x_2(t)$ for $x'' + a_1(t)x' + a_0(t) = 0$
  The general solution $x(t) = x_h(t) + x_p(x)$, where $x_h(t) = c_1 x_1(t) + c_2 x_2(t)$
  variation of parameters: the particular solution $x_p(t) = c_1(t)x_1(t) + c_2(t)x_2(t)$ satisfies
  $$\begin{pmatrix} x_1 & x_2 \\ x_1' & x_2' \end{pmatrix}\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$
  Thus, use Cramer's rule and integrate, where $W(t) \equiv \left|\begin{smallmatrix} x_1 & x_2 \\ x_1' & x_2' \end{smallmatrix}\right| = W(t_0)e^{-\int_{t_0}^{t} a_1(\tau)d\tau}$
  $$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} -\int \frac{f(t)u_2(t)}{W(t)}dt \\ \int \frac{f(t)u_1(t)}{W(t)}dt \end{pmatrix}$$
- non-homogeneous, special function $f(t)$, constant coefficients $x'' + a_1 x' + a_0 x = f(t)$
  undetermined coefficients:
  $f(t)$ are α, $e^{\beta t}$, $\sin(\omega t)$, $\cos(\omega t)$, t^n, and sums, products of these common functions

Form of source function $f(t)$	Trial form of particular solution $x_p(t)$
α	A
$\alpha e^{\beta t}$	$Ae^{\beta t}$
Polynomial of degree n	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$
$\alpha \sin \omega t; \alpha \cos \omega t$	$A \sin \omega t + B \cos \omega t$
$\alpha e^{rt} \sin \omega t; \quad \alpha e^{rt} \cos \omega t$	$e^{rt}(A \sin \omega t + B \cos \omega t)$

(2) Cauchy-Euler equation $t^2 x'' + b_1 t x' + b_0 x = 0$
The **substitution** $p = \ln(t)$, it leads to
$$dx = x'dt = x'\frac{dt}{dp}dp = x'tdp$$
$$d^2x = x''dt^2 = d(dx) = d\left(x'tdp\right) = d\left(x't\right)dp = \left(\left[x''\frac{dt}{dp}dp\right]t + x'\frac{dt}{dp}dp\right)dp = \left[x''t^2 + x't\right]dp^2$$

Thus

$$\frac{dx}{dp} = tx', \quad \frac{d^2x}{dp^2} - \frac{dx}{dp} = t^2 x''$$

Reduce to the homogeneous, constant coefficient equation

$$\frac{d^2x}{dp^2} + (b_1 - 1)\frac{dx}{dp} + b_0 x = 0$$

The **indicial equation**

$$r(r-1) + b_1 r + b_0 = r^2 + (b_1 - 1)r + b_0 = 0$$

(3) nonlinear equation

- special form $x'' = f(t, x')$
  The **substitution** $v \equiv x'$, it becomes first order equation

$$\frac{dv}{dt} = f(t, v)$$

- special form $x'' = f(x, x')$
  The **substitution** $v \equiv x'$, it leads to

$$x'' = \frac{dx'}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

  Reduce to the first order equation

$$v\frac{dv}{dx} = f(x, v)$$

Higher order equations

(1) linear equations

- homogeneous equation, constant coefficients $x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_0 x = 0$
  The **characteristic equation**

$$r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0$$

  It has m roots $r_1, r_2, \cdots, r_m$, the multiplicity of which, respectively, is equal to $k_1, k_2, \cdots, k_m$

$$\begin{aligned} x(t) = &\left(c_1 + c_2 t + \cdots + c_{k_1} t^{k_1 - 1}\right) e^{r_1 t} + \cdots \\ &+ \left(c_{n-k_m+1} + c_{n-k_m+2} t + \cdots + c_n t^{k_m - 1}\right) e^{r_m t} \end{aligned}$$

- non-homogeneous equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = f(t)$
  Linear independent $x_1(t), \cdots, x_n(t)$ known for $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$
  The general solution $x(t) = x_h(t) + x_p(x)$, where $x_h(t) = \sum_{i=1}^n c_i x_i(t)$
  variation of parameters: the particular solution $x_p(t) = \sum_{i=1}^n c_i(t) x_i(t)$ satisfies

$$\begin{pmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t) & x_2^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{pmatrix} \begin{pmatrix} c_1'(t) \\ c_2'(t) \\ \vdots \\ c_n'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{pmatrix}$$

Thus, use Cramer's rule and integrate, where Wronskian $W(t) = W(t_0)e^{-\int_{t_0}^{t} a_{n-1}(\tau)d\tau}$, and $W_i(t)$ is the Wronskian determinant with the i-th column replaced by $\begin{bmatrix} 0 & 0 & \cdots & f(t) \end{bmatrix}^T$

$$\begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix} = \int \begin{pmatrix} \frac{W_1(t)}{W(t)} \\ \frac{W_2(t)}{W(t)} \\ \vdots \\ \frac{W_n(t)}{W(t)} \end{pmatrix} dt$$

- non-homogeneous, special $f(t)$, constant coefficients $x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_0 x = f(t)$
  undetermined coefficients:
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