ECE600 Computer Problem (Extra Credit)

Assigned 11/18/21

- Email a pdf of your derivations, plots, and code to Prof Gelfand
- Due 12/10/21 (or earlier)

1. Let $X_1, X_2, \ldots,$ be independent uniform random variables on $(0, 1)$. Let

$$S_n = \sum_{i=1}^{n} X_i$$

and

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

where $\mathrm{E}[X_n] = \mu$ and $\mathrm{Var}[X_n] = \sigma^2$.

   (a) Find μ and σ^2.

   (b) Show that $\mathrm{E}[Z_n] = 0$ and $\mathrm{Var}[Z_n] = 1$.

   (c) Find and plot $f_{Z_3}(z)$.

2. For $n > 3$ $f_{Z_n}(z)$ can be approximated analytically or numerically.

An analytical approximation for $f_{Z_n}(z)$ can be based on the Central Limit Theorem, i.e., a Gaussian pdf with mean 0 and variance 1 can be used to estimate the pdf of Z_n.

A numerical approximation for $f_{Z_n}(z)$ can be based on a simulation. Generate $n \times K$ realizations of a uniform random variable on $(0, 1)$, and denote them by $\{x_{i,k} : i = 1, \ldots, n; k = 1, \ldots, K\}$. $\{x_{i,k} : k = 1, \ldots, K\}$ are K realizations of X_i. Let

$$s_{n,k} = \sum_{i=1}^{n} x_{i,k}, \quad z_{n,k} = \frac{s_{n,k} - n\mu}{\sigma\sqrt{n}}.$$

$\{z_{n,k} : k = 1, \ldots, K\}$ are K realizations of Z_n. A normalized histogram for Z_n based on these realizations can be used to estimate the pdf of Z_n.

For $n = 3$ plot and compare the analytical and numerical approximations with the exact plot in part 1. For $n = 30$ plot and compare the analytical and numerical approximations with each other. In the simulations, choose $K = 10^6$ and a bin size of 100.

3. To see how the approximations might be used, consider the following problem. Suppose n messages arrive at a node where the message lengths are independent uniform random variables between 0 and 1 MB (this is an approximations as bits are of course discrete). These messages are to be stored on a hard drive or transmitted over a network. For $n = 3$ find the probability that the total message length exceeds 2 MB using the analytical and numerical approximations in part 2, and also the exact probabibility using part 1. For $n = 30$ find the probability that the total message length exceeds 20 MB using the analytical and numerical approximations in part 2.