# ECE 662: Homework 2

by

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Question

Solution

If $\phi$ is **NOT injective**, cannot find $\psi$ to define a metric $d_P$ on $P$

If $\phi$ is **injective**, can find $\psi$ to define a metric $d_P$ on $P$

## Question

There exists a map $\phi : P \to S$, for $\forall p \in P$ and $x \in S$

$$\phi : p \mapsto x = \phi(p)$$

We also define the inner product $\cdot : S \times S \to \mathbb{R}$, which satisfies

- $\forall x, x' \in S, x \cdot x' = x' \cdot x$
- $\forall x \in S, x \cdot x \geq 0$ and $x \cdot x = 0 \Leftrightarrow x = \mathbf{0} \in S$
- $\forall x, x', x'' \in S, \forall a, b \in \mathbb{R}, (ax + bx') \cdot x'' = a(x \cdot x'') + b(x' \cdot x'')$

Thus, we can define the kernel function $K : P \times P \to \mathbb{R}_{\geq 0}$, for $\forall p, p' \in P$

$$K(p, p') := \phi(p) \cdot \phi(p')$$

$S$ is a Hilbert space with a metric $d : S \times S \to \mathbb{R}_{\geq 0}$ defined based on our inner product, for $\forall x, x' \in S$

$$d(x, x') := \sqrt{(x - x') \cdot (x - x')}$$

which must satisfy

- $\forall x, x' \in S, d(x, x') = d(x', x)$
- $\forall x, x' \in S, d(x, x') \geq 0$ and $d(x, x') = 0 \Leftrightarrow x = x'$
- $\forall x, x', x'' \in S, d(x, x') + d(x', x'') \geq d(x, x'')$

The 1st/2nd properties of a metric $d$ can be proved with the 1st/2nd properties of our inner product. The 3rd property (distance inequality) above can be proved using the 3rd property of our inner product and Cauchy-Schwarz inequlity
$((x - x') \cdot (x - x'))((x' - x'') \cdot (x' - x'')) \geq ((x - x') \cdot (x' - x''))^2$

**Question**: Does this meric $d : S \times S \to \mathbb{R}_{\geq 0}$ on $S$ induce a metric $d_P : P \times P \to \mathbb{R}_{\geq 0}$ on $P$?

More specifically, can we define a metric $d_P : P \times P \to \mathbb{R}_{\geq 0}$ on $P$ based on a function $\psi : \phi[P] \times \phi[P] \to \mathbb{R}_{\geq 0}$, where $\phi[P] \subset S$?

$$d_P(p, p') := \psi(\phi(p), \phi(p')) \quad \forall p, p' \in P$$

## Solution

If we can find a metric $d_P : P \times P \to \mathbb{R}_{\geq 0}$ on $P$ based on a function $\psi : \phi[P] \times \phi[P] \to \mathbb{R}_{\geq 0}$, where $\phi[P] \subset S$

$$d_P(p, p') := \psi(\phi(p), \phi(p')) \quad \forall p, p' \in P$$

Let's list the requirements for $d_P : P \times P \to \mathbb{R}_{\geq 0}$

- $\forall p, p' \in P, d_P(p, p') = d_P(p', p)$
- $\forall p, p' \in P, d_P(p, p') \geq 0$ and $d_P(p, p') = 0 \Leftrightarrow p = p'$
- $\forall p, p', p'' \in P, d_P(p, p') + d_P(p', p'') \geq d_P(p, p'')$

Thus, for $\psi : \phi[P] \times \phi[P] \to \mathbb{R}_{\geq 0}$, it must satisfy

- $\forall x, x' \in \phi[P] \subset S, \psi(x, x') = \psi(x', x)$
- $\forall x, x' \in \phi[P] \subset S, \psi(x, x') \geq 0$ and $\psi(x, x') = 0 \Leftrightarrow x = x'$
- $\forall x, x', x'' \in \phi[P] \subset S, \psi(x, x') + \psi(x', x'') \geq \psi(x, x'')$

## If $\phi$ is **NOT injective**, cannot find $\psi$ to define a metric $d_P$ on $P$

When $\phi : P \to S$ is **NOT** a one-to-one function

$$\exists p, p' \in P, p \neq p', \quad \text{s.t. } \phi(p) = \phi(p')$$

Denote $x_0 := \phi(p) = \phi(p') \in \phi[P] \subset S$, with the 2nd property of the metric $d_P$ and $\psi$

$$d_P(p, p') = \psi(\phi(p), \phi(p')) = \psi(x_0, x_0) = 0 \Longrightarrow p = p'$$

This leads to a contradiction, thus we **cannot** find $\psi : \phi[P] \times \phi[P] \to \mathbb{R}_{\geq 0}$ to define a metric $d_P$ on $P$

$$d_P(p, p') := \psi(\phi(p), \phi(p')) \quad \forall p, p' \in P$$

## If $\phi$ is **injective**, can find $\psi$ to define a metric $d_P$ on $P$

When $\phi : P \to S$ is a one-to-one function

$$\forall p, p' \in P, \quad \text{s.t. } \phi(p) = \phi(p') \Leftrightarrow p = p'$$

We can define $\psi : \phi[P] \times \phi[P] \to \mathbb{R}_{\geq 0}$ based on our inner product $\cdot : S \times S \to \mathbb{R}$, which satisfies

- $\forall x, x' \in S, x \cdot x' = x' \cdot x$
- $\forall x \in S, x \cdot x \geq 0$ and $x \cdot x = 0 \Leftrightarrow x = \mathbf{0} \in S$
- $\forall x, x', x'' \in S, \forall a, b \in \mathbb{R}, (ax + bx') \cdot x'' = a(x \cdot x'') + b(x' \cdot x'')$

$\psi : \phi[P] \times \phi[P] \to \mathbb{R}_{\geq 0}$ is defined as below

$$\psi(x, x') := \sqrt{(x - x') \cdot (x - x')}, \quad \forall x, x' \in \phi[P] \subset S$$

With the 1st/3rd properties of our inner product and notice the kernel function $K : P \times P \to \mathbb{R}_{\geq 0}$, for $\forall p, p' \in P, K(p, p') := \phi(p) \cdot \phi(p')$, we can rewrite $d_P : P \times P \to \mathbb{R}_{\geq 0}$ as

$$\begin{aligned} d_P(p, p') :=& \psi(\phi(p), \phi(p')) = \sqrt{(\phi(p) - \phi(p')) \cdot (\phi(p) - \phi(p'))} \\ =& \sqrt{K(p, p) + K(p', p') - 2K(p, p')} \end{aligned}$$

Let's verify the three properties for $d_P : P \times P \to \mathbb{R}_{\geq 0}$

- $\forall p, p' \in P, d_P(p, p') = d_P(p', p)$
- $\forall p, p' \in P, d_P(p, p') \geq 0$ and $d_P(p, p') = 0 \Leftrightarrow p = p'$
- $\forall p, p', p'' \in P, d_P(p, p') + d_P(p', p'') \geq d_P(p, p'')$

For 1st property, it is equivalent to $\phi(p) \cdot \phi(p') = \phi(p') \cdot \phi(p)$, thus it can be proved with the 1st property of our inner product

For 2nd property, since $\phi(p), \phi(p') \in \phi[P] \subset S \Rightarrow \phi(p) - \phi(p') \in S,$ thus it can be proved with the 2nd property of our inner product

The 3rd property (distance inequality) above can be proved using the 3rd property of our inner product and Cauchy-Schwarz inequlity

$$\begin{aligned} &[(\phi(p) - \phi(p')) \cdot (\phi(p) - \phi(p'))][(\phi(p') - \phi(p'')) \cdot (\phi(p') - \phi(p''))] \\ &\geq [(\phi(p) - \phi(p')) \cdot (\phi(p') - \phi(p''))]^2 \end{aligned}$$