Zhankun Luo (ECE Ph.D. student)

Problem 1

1. Let Y be a random variable which is uniformly distributed on the interval $[0,1]$; so that it has a PDF of $p_Y(y) = 1$ for $0 \le y \le 1$ and 0 otherwise. Calculate the CDF of Y and use this to calculate both the CDF and PDF of $Z = Y^2$.

solution

Here we denote the PDF of Y by $f_Y(y)$ instead of $p_Y(y)$. The CDF of y is

$$F_Y(z) = \int_{-\infty}^{y} f_Y(y')\mathrm{d}y' = \begin{cases} 0 & \text{for } y < 0 \\ y & \text{for } 0 \le y \le 1 \\ 1 & \text{for } 1 < y \end{cases}$$

With the **Jacobian formula**: if $Z = g(Y)$ is a strictly monotone function of Y with a nonzero derivative $\frac{\mathrm{d}Z}{\mathrm{d}Y} \neq 0$. Then Z has pdf

$$f_Z(z) = f_Y\left(g^{-1}(z)\right)\left|\frac{\mathrm{d}g^{-1}(z)}{\mathrm{d}z}\right| = \frac{f_Y\left(g^{-1}(z)\right)}{\left|\frac{\mathrm{d}g(y)}{\mathrm{d}y}\Big|_{y=g^{-1}(z)}\right|}$$

Notice that $g(Y) = Y^2$ and $Z = g(Y) \in [0,1]$ strictly increases for $Y \in [0,1]$

$$f_Z(z) = \frac{f_Y\left(g^{-1}(z)\right)}{\left|\frac{\mathrm{d}g(y)}{\mathrm{d}y}\Big|_{y=g^{-1}(z)}\right|} \quad \text{for } z \in [0,1]$$

Where the inverse function $y = g^{-1}(z) \in [0,1], z \in [0,1]$ is defined by

$$y = g^{-1}(z) = \sqrt{z}, \quad \frac{\mathrm{d}g(y)}{\mathrm{d}y}\Big|_{y=g^{-1}(y)} = 2y|_{y=\sqrt{z}} = 2\sqrt{z}$$

Thus, we conclude that the PDF $f_Z(z)$ of Z is

$$f_Z(z) = \begin{cases} \frac{1}{2\sqrt{z}} & \text{for } 0 \le z \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

So, the CDF $F_Z(z)$ of Z is

$$F_Z(z) = \int_{-\infty}^{z} f_Z(z')\mathrm{d}z' = \begin{cases} 0 & \text{for } z < 0 \\ \sqrt{z} & \text{for } 0 \le z \le 1 \\ 1 & \text{for } 1 < z \end{cases}$$

Problem 2

2. In this problem, we present a method for generating a random variable, Y, with any valid CDF specified by $F_Y(t)$. To do this, let X be a uniformly distributed random variable on the interval $[0,1]$, and let $Y = f(X)$ where we define the function

$$f(u) = \inf\{t \in \mathbb{R} \mid F_Y(t) \geq u\}$$

Prove that in the general case, Y has the desired CDF.
(Hint: First show that the following two sets are equal, $(-\infty, F_Y(t)] = \{u \in \mathbb{R} : f(u) \leq t\}$.)

solution
We can find details of proof in the PDF chapter [1] and the lecture note [2]
Apply $F_Y(t)$ on both sides of an inequality $f(u) = \inf\{t' \in \mathbb{R} \mid F_Y(t') \geq u\} \leq t$
Notice that $F_Y(t)$ is non-decreasing, and right-continuous $f(u) \in \{t' \in \mathbb{R} \mid u \leq F_Y(t')\}$

$$f(u) = \inf\{t' \in \mathbb{R} \mid u \leq F_Y(t')\} \leq t \quad \Rightarrow \quad u \leq F_Y(f(u)) \leq F_Y(t)$$

It leads to the relationship between two events

$$\{f(u) \leq t\} \subseteq \{u \leq F_Y(t)\} \tag{1}$$

On the other hand, $u < F_Y(t)$ leads to

$$u < F_Y(t) \quad \Rightarrow \quad t \in \{t' \in \mathbb{R} \mid u \leq F_Y(t')\} \quad \Rightarrow \quad f(u) = \inf\{t' \in \mathbb{R} \mid u \leq F_Y(t')\} \leq t$$

It leads to the other relationship between two events

$$\{u < F_Y(t)\} \subseteq \{f(u) \leq t\} \tag{2}$$

Combine (1), (2) and replace u with the random variable X, and notice that $Y = f(X)$

$$\{X < F_Y(t)\} \subseteq \{f(X) \leq t\} = \{Y \leq t\} \subseteq \{X \leq F_Y(t)\}$$

Thus, for the probabilities Pr of events

$$\Pr\{X < F_Y(t)\} \leq \Pr\{Y \leq t\} \leq \Pr\{X \leq F_Y(t)\}$$

Since $X \sim U(0,1)$, then X is a continuous random variable, so $\Pr\{X = F_Y(t)\} = 0$

$$\Pr\{X < F_Y(t)\} = \Pr\{Y \leq t\} = \Pr\{X \leq F_Y(t)\}$$

Since $X \sim U(0,1)$, we can write down PDF $f_X(x)$ of X

$$f_X(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

As a specified valid CDF, $0 \leq F_Y(t) \leq 1$

$$\Pr\{Y \leq t\} = \Pr\{X \leq F_Y(t)\} = \int_{-\infty}^{F_Y(t)} f_X(x)\mathrm{d}x = \int_0^{F_Y(t)} 1\mathrm{d}x = F_Y(t)$$

In the end, we prove that Y has the desired CDF $F_Y(t)$

References

[1] Luc Devroye (1986). "Section 2.2. Inversion by numerical solution of F(X) = U" (PDF). *Non-Uniform Random Variate Generation.* New York: Springer-Verlag.
[2] Karl Sigman (2010). "1 Inverse Transform Method" (PDF). *Professor Karl Sigman's Lecture Notes on Simulation.* New York: Columbia University.

Problem 3

3. Give an example of three random variables, X_1, X_2, and X_3 such that for $k = 1, 2, 3, P\{X_k = 1\} = P\{X_k = -1\} = \frac{1}{2}$, and such X_k and X_j are independent for all $k \neq j$, but such that X_1, X_2, and X_3 are not jointly independent.

solution

We may suppose that there are two independent random variable $\alpha_1, \alpha_2 \in \{0, 1\}$

$$P\{\alpha_1 = 0\} = P\{\alpha_2 = 0\} = P\{\alpha_1 = 1\} = P\{\alpha_2 = 1\} = \frac{1}{2}$$

$$P\{\alpha_1 = 0, \alpha_2 = 0\} = P\{\alpha_1 = 0\} \cdot P\{\alpha_2 = 0\}$$

$$P\{\alpha_1 = 0, \alpha_2 = 1\} = P\{\alpha_1 = 0\} \cdot P\{\alpha_2 = 1\}$$

$$P\{\alpha_1 = 1, \alpha_2 = 0\} = P\{\alpha_1 = 1\} \cdot P\{\alpha_2 = 0\}$$

$$P\{\alpha_1 = 1, \alpha_2 = 1\} = P\{\alpha_1 = 1\} \cdot P\{\alpha_2 = 1\}$$

And define the X_1, X_2, X_3 as follows

$$X_1 \equiv (-1)^{\alpha_1}, \quad X_2 \equiv (-1)^{\alpha_2}, \quad X_3 \equiv (-1)^{\alpha_1 + \alpha_2}$$

Then, it is easy to be verified

$$P\{X_k = 1\} = P\{X_k = -1\} = \frac{1}{2} \quad \text{for } k = 1, 2, 3$$

X_k and X_j are independent for all $k \neq j$

For X_1, X_2, we have

$$P\{X_1 = 1, X_2 = 1\} = P\{\alpha_1 = 0, \alpha_2 = 0\} = P\{\alpha_1 = 0\} \cdot P\{\alpha_2 = 0\} = P\{X_1 = 1\} \cdot P\{X_2 = 1\}$$

$$P\{X_1 = 1, X_2 = -1\} = P\{\alpha_1 = 0, \alpha_2 = 1\} = P\{\alpha_1 = 0\} \cdot P\{\alpha_2 = 1\} = P\{X_1 = 1\} \cdot P\{X_2 = -1\}$$

$$P\{X_1 = -1, X_2 = 1\} = P\{\alpha_1 = 1, \alpha_2 = 0\} = P\{\alpha_1 = 1\} \cdot P\{\alpha_2 = 0\} = P\{X_1 = -1\} \cdot P\{X_2 = 1\}$$

$$P\{X_1 = -1, X_2 = -1\} = P\{\alpha_1 = 1, \alpha_2 = 1\} = P\{\alpha_1 = 1\} \cdot P\{\alpha_2 = 1\} = P\{X_1 = -1\} \cdot P\{X_2 = -1\}$$

Similarly, we can verify the independence for X_1, X_3 and X_2, X_3 as well

But we know that

$$P\{X_1 = -1, X_2 = -1, X_3 = -1\} = P\{\alpha_1 = 1, \alpha_2 = 1, (\alpha_1 + \alpha_2) \equiv 1 \bmod(2)\} = 0$$

$$\neq P\{X_1 = -1\}P\{X_2 = -1\}P\{X_3 = -1\} = P\{\alpha_1 = 1\}P\{\alpha_2 = 1\}P\{(\alpha_1 + \alpha_2) \equiv 1 \bmod(2)\} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

So, X_1, X_2, and X_3 are not jointly independent

Problem 4

4. Let X be a jointly Gaussian random vector, and let $A \in \mathbb{R}^{M\times N}$ be a rank M matrix. Then prove that the vector $Y = AX$ is also jointly Gaussian.

solution

Since $M = \text{rank}(A) \le \min(M, N)$, we have $M \le N$

Since X is a jointly Gaussian random vector, the PDF $f_X(x)$ of X is given by

$$f_X(x) = \frac{1}{(2\pi)^{N/2}}|R|^{-1/2}\exp\left\{-\frac{1}{2}(x-\mu)^T R^{-1}(x-\mu)\right\}$$

Where the mean vector μ, and symmetric positive-definite covariance $R = R^T$ is given by

$$\mathbb{E}[X] = \mu$$
$$\mathbb{E}\left[(X-\mu)(X-\mu)^T\right] = R$$

Let's write R in such a form, where E is an orthonomal matrix($E^T = E^{-1}$) whose columns are the eigenvectors of R and $\Lambda = \text{diag}(\sigma_1^2, \cdots, \sigma_N^2)$ is a diagonal matrix of strictly positive eigenvalues

$$R = E\Lambda E^T$$

Define the decorrelated vector $\tilde{X}$ as

$$\tilde{X} = E^T X$$

Write $AE = [\alpha_1, \cdots, \alpha_M]^T$ in such a form of stacking M row vectors $\alpha_k^T, k = 1, \cdots, M$ together

$$Y = (y_1, \cdots, y_M)^T = AX = AE\tilde{X} = [\alpha_1, \cdots, \alpha_M]^T\tilde{X} = [\alpha_1^T\tilde{X}, \cdots, \alpha_M^T\tilde{X}]^T$$

Thus, with the transformation $\tilde{X} = E^T X = (\tilde{x}_1, \cdots, \tilde{x}_N)^T, \tilde{\mu} = E^T\mu = (\tilde{\mu}_1, \cdots, \tilde{\mu}_N)^T$, then the corresponding PDF is

$$f_{\tilde{X}}(\tilde{x}) = \left(\frac{1}{2\pi}\right)^{N/2}\frac{1}{\prod_{k=1}^N \sigma_k}\exp\left\{-\sum_{k=1}^{n}\frac{(\tilde{x}_k - \tilde{\mu}_k)^2}{2\sigma_k^2}\right\}$$

And notice that $\text{rank}(AE) = M$

$$\mathbb{E}[\tilde{X}] = \tilde{\mu}$$
$$\mathbb{E}\left[(\tilde{X}-\tilde{\mu})(\tilde{X}-\tilde{\mu})^T\right] = \Lambda$$

Let's introduce the **characteristic function** of $Y = (y_1, \cdots, y_M)^T = AX = [\alpha_1^T \tilde{X}, \cdots, \alpha_M^T \tilde{X}]^T$
$AE = [\beta_1, \cdots, \beta_N]$ also can be written in the form of stacking column vectors

$$\begin{aligned}
\Phi_Y(\omega) &= \mathbb{E}\left(e^{j\omega^T y}\right) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_Y(y) e^{j\omega^T y} \mathrm{d}y_1 \cdots \mathrm{d}y_M \\
&= \mathbb{E}\left(e^{j\omega^T AE\tilde{x}}\right) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_{\tilde{X}}(\tilde{x}) e^{j\omega^T AE\tilde{x}} \mathrm{d}x_1 \cdots \mathrm{d}x_N \\
&= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi}\right)^{N/2} \frac{1}{\prod_{k=1}^N \sigma_k} \exp\left\{-\sum_{k=1}^n \frac{(\tilde{x}_k - \tilde{\mu}_k)^2}{2\sigma_k^2}\right\} e^{j\omega^T AE\tilde{x}} \mathrm{d}\tilde{x}_1 \cdots \mathrm{d}\tilde{x}_N \\
&= \prod_{k=1}^N \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left\{-\frac{(\tilde{x}_k - \tilde{\mu}_k)^2}{2\sigma_k^2} + j(\omega^T \beta_k)\tilde{x}_k\right\} \mathrm{d}\tilde{x}_k \\
&= \prod_{k=1}^N \exp\left\{-\frac{(\omega^T\beta_k)^2\sigma_k^2}{2} + j(\omega^T\beta_k)\tilde{\mu}_k\right\} \cdot \prod_{k=1}^N \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left(\frac{\tilde{x}_k}{\sigma_k} - \frac{\tilde{\mu}_k}{\sigma_k} - j(\omega^T\beta_k)\sigma_k\right)^2}{2}\right\} \mathrm{d}\left(\frac{\tilde{x}_k}{\sigma_k}\right) \\
&= \prod_{k=1}^N \exp\left\{-\frac{(\omega^T\beta_k)^2\sigma_k^2}{2} + j(\omega^T\beta_k)\tilde{\mu}_k\right\} \\
&= \exp\left(-\frac{1}{2}\omega^T AE\Lambda E^T A^T \omega + j\omega^T AE\tilde{\mu}\right) = \exp\left(-\frac{1}{2}\omega^T ARA^T \omega + j\omega^T A\mu\right)
\end{aligned}$$

Then, we can do the inverse transform to compute the PDF $f_Y(y)$.
Since, $\mathrm{Rank}(ARA^T) = M$, with the decomposition $[ARA] = VDV^T$, we can define the immediate variable $\bar{\omega} = V^T\omega$ has the Jacobian $\left|\frac{\mathrm{d}\omega}{\mathrm{d}\bar{\omega}}\right| = |V| = 1$ and $\bar{y} = V^T y, \nu = V^T A\mu$, where diagonal matrix $D = \mathrm{diag}(d_1^2, \cdots, d_M^2)$ and the orthonomal matrix V has $V^T = V^{-1}$
Notice that $|ARA^T| = |D| = (\prod_{k=1}^M d_k)^2, (ARA^T)^{-1} = VD^{-1}V^T$

$$\begin{aligned}
f_Y(y) &= \frac{1}{(2\pi)^M} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \Phi_Y(\omega) e^{-j\omega^T y} \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_M \\
&= \frac{1}{(2\pi)^M} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\omega^T ARA^T\omega + j\omega^T A\mu\right) e^{-j\omega^T y} \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_M \\
&= \frac{1}{(2\pi)^M} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\bar{\omega}^T D\bar{\omega} + j\bar{\omega}^T\nu\right) e^{-j\bar{\omega}^T\bar{y}} \left|\frac{\mathrm{d}\omega}{\mathrm{d}\bar{\omega}}\right| \mathrm{d}\bar{\omega}_1 \cdots \mathrm{d}\bar{\omega}_M \\
&= \frac{1}{(2\pi)^{M/2}} \cdot \frac{1}{\prod_{k=1}^M d_k} \cdot \prod_{k=1}^M \int_{-\infty}^{+\infty} \frac{d_k}{\sqrt{2\pi}} \exp\left\{-\frac{d_k^2\bar{\omega}_k^2}{2} - j(\bar{y}_k - \nu_k)\bar{\omega}_k\right\} \mathrm{d}\bar{\omega}_k \\
&= \frac{1}{(2\pi)^{M/2}} \cdot \frac{1}{\prod_{k=1}^M d_k} \cdot \prod_{k=1}^M \exp\left\{-\frac{(\bar{y}_k - \nu_k)^2}{2d_k^2}\right\} \cdot \prod_{k=1}^M \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left(d_k\bar{\omega}_k + j\frac{(\bar{y}_k - \nu_k)}{d_k}\right)^2}{2}\right\} \mathrm{d}\left(d_k\bar{\omega}_k\right) \\
&= \frac{1}{(2\pi)^{M/2}} \cdot \frac{1}{\prod_{k=1}^M d_k} \cdot \prod_{k=1}^M \exp\left\{-\frac{(\bar{y}_k - \nu_k)^2}{2d_k^2}\right\} = \frac{1}{(2\pi)^{M/2}} |D|^{-1/2} \exp\left\{-\frac{1}{2}(\bar{y} - \nu)^T D^{-1}(\bar{y} - \nu)\right\} \\
&= \frac{1}{(2\pi)^{M/2}} |ARA^T|^{-1/2} \exp\left\{-\frac{1}{2}(y - A\mu)^T (ARA^T)^{-1}(y - A\mu)\right\}
\end{aligned}$$

So, we prove that the vector $Y = AX$ is also jointly Gaussian.

Problem 5

5. Let $X \sim N(0, R)$ where R is a $p \times p$ symmetric positive-definite matrix.

a) Prove that if for all $i \neq j, \mathbb{E}[X_i X_j] = 0$ (i.e., X_i and X_j are uncorrelated), then X_i and X_j are pair-wise independent.
b) Prove that if for all i, j, X_i and X_j are uncorrelated, then the components of X are jointly independent.

solution

a) Because all i, j, X_i and X_j are uncorrelated, we have $R_{ij} = 0$ for $i \neq j$, denote $\mathbb{E}\left[X_k^2\right] = \sigma_k^2$ for $k \in \{1, \cdots, p\}$

$$R = \operatorname{diag}\left(\sigma_1^2, \cdots, \sigma_p^2\right), \quad R^{-1} = \operatorname{diag}\left(\frac{1}{\sigma_1^2}, \cdots, \frac{1}{\sigma_p^2}\right), \quad |R|^{-1/2} = \prod_{k=1}^{p} \frac{1}{\sigma_k}$$

Thus, for the PDF $f_X(x)$ of X

$$f_X(x_1, \cdots, x_p) = \prod_{k=1}^{p} \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left\{-\frac{x_k^2}{2\sigma_k^2}\right\}$$

Notice that for $k \in \{1, \cdots, p\}$

$$f_X(x_k) = \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left\{-\frac{x_k^2}{2\sigma_k^2}\right\} \cdot \prod_{k'=1, k' \neq k}^{p} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_{k'}} \exp\left\{-\frac{x_{k'}^2}{2\sigma_{k'}^2}\right\} \mathrm{d}x_{k'} = \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left\{-\frac{x_k^2}{2\sigma_k^2}\right\}$$

For $X_i, X_j, i \neq j$, we have

$$\begin{aligned} f_X(x_i, x_j) &= \frac{1}{2\pi\sigma_i\sigma_j} \exp\left\{-\frac{x_i^2}{2\sigma_i^2} - \frac{x_j^2}{2\sigma_j^2}\right\} \cdot \prod_{k'=1, k' \neq i, j}^{p} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_{k'}} \exp\left\{-\frac{x_{k'}^2}{2\sigma_{k'}^2}\right\} \mathrm{d}x_{k'} \\ &= \frac{1}{2\pi\sigma_i\sigma_j} \exp\left\{-\frac{x_i^2}{2\sigma_i^2} - \frac{x_j^2}{2\sigma_j^2}\right\} \\ &= f_X(x_i) \cdot f_X(x_j) \end{aligned}$$

So, we prove that if for all $i \neq j, \mathbb{E}[X_i X_j] = 0$ (i.e., X_i and X_j are uncorrelated), then X_i and X_j are pair-wise independent.

b) Similarly, we conclude

$$f_X(x_1, \cdots, x_p) = \prod_{k=1}^{p} \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left\{-\frac{x_k^2}{2\sigma_k^2}\right\} = \prod_{k=1}^{p} f_X(x_k)$$

In the end, we prove that if for all i, j, X_i and X_j are uncorrelated, then the components of X are jointly independent.