# Homework 5

Course Title: Digital Signal Processing I (Spring 2020)

Course Number: ECE53800

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Problems 6.1, 6.4, 6.5, 6.6, 6.9, 6.11, 6.15, 6.16, 6.18, 6.25 (a), (b)

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# Problems

## Problem 6.1

Given a difference equation

$$y(n) = x(n) - 0.5y(n-1)$$

1. Calculate the system response y(n) for n = 0, 1, ..., 4 with the input x(n) = $(0.5)^n$u(n) and initial condition: y(-1) = 1.
2. Calculate the system response y(n) for n = 0, 1, ..., 4 with the input x(n) = $(0.5)^n$u(n) and zero initial condition: y(-1) = 0.

### solution

1. y(-1) = 1

$$\begin{aligned}
y(0) &= x(0) - 0.5y(0-1) = 0.5^0 - 0.5\times 1 = 0.5 \\
y(1) &= x(1) - 0.5y(1-1) = 0.5^1 - 0.5\times 0.5 = 0.5^2 \\
y(2) &= x(2) - 0.5y(2-1) = 0.5^2 - 0.5\times 0.5^2 = 0.5^3 \\
y(3) &= x(3) - 0.5y(3-1) = 0.5^3 - 0.5\times 0.5^3 = 0.5^4 \\
y(4) &= x(4) - 0.5y(4-1) = 0.5^4 - 0.5\times 0.5^4 = 0.5^5
\end{aligned}$$

Thus, y(n) for n = 0, 1, ..., 4 = [0.5, 0.25, 0.125, 0.0625, 0.03125]

2. y(-1) = 0

$$\begin{aligned}
y(0) &= x(0) - 0.5y(0-1) = 0.5^0 - 0.5\times 0 = 0.5^0 \\
y(1) &= x(1) - 0.5y(1-1) = 0.5^1 - 0.5\times 0.5^0 = 0 \\
y(2) &= x(2) - 0.5y(2-1) = 0.5^2 - 0.5\times 0 = 0.5^2 \\
y(3) &= x(3) - 0.5y(3-1) = 0.5^3 - 0.5\times 0.5^2 = 0 \\
y(4) &= x(4) - 0.5y(4-1) = 0.5^4 - 0.5\times 0 = 0.5^4
\end{aligned}$$

Thus, y(n) for n = 0, 1, ..., 4 = [1, 0, 0.25, 0, 0.0625]

## Problem 6.4

Given the following difference equation

$$y(n) = 0.5x(n) + 0.5x(n-1)$$

sampled at a rate of 8,000 Hz,

1. Find H(z).
2. Determine the impulse response y(n) if the input x(n)=4δ(n).
3. Determine the step response y(n) if the input x(n)=10u(n).

### solution

1. Find H(z).

$$\begin{aligned}\sum_{n=0}^{\infty} y(n)z^{-n} &= \sum_{n=0}^{\infty}[0.5x(n) + 0.5x(n-1)]z^{-n} \\ Y(z) &= 0.5X(z) + 0.5z^{-1}[x(-1)z + X(z)] \quad \{x(-1) = 0\} \\ Y(z) &= [0.5 + 0.5z^{-1}]X(z) \\ H(z) &\equiv \frac{Y(z)}{X(z)} = 0.5 + 0.5z^{-1}\end{aligned}$$

Thus $H(z) = 0.5 + 0.5z^{-1}$

2. x(n)=4δ(n), find y(n)

$$\begin{aligned}X(z) &= \sum_{n=0}^{\infty} 4\delta(n)z^{-n} = 4 \\ Y(z) &= H(z)\cdot X(z) = [0.5 + 0.5z^{-1}]\cdot 4 = 2 + 2z^{-1} \\ y(n) &= \boldsymbol{Z}^{-1}[Y(z)] = 2\delta(n) + 2\delta(n-1)\end{aligned}$$

Thus the impulse response y(n)=2δ(n)+2δ(n-1)

3. x(n)=10u(n), find y(n)

$$\begin{aligned}X(z) &= \sum_{n=0}^{\infty} 10\mathrm{u}(n)z^{-n} = \frac{10}{1-z^{-1}} \\ Y(z) &= H(z)\cdot X(z) = [0.5 + 0.5z^{-1}]\cdot \frac{10}{1-z^{-1}} = 5\frac{1+z^{-1}}{1-z^{-1}} = 5[-1 + \frac{2}{1-z^{-1}}] \\ y(n) &= \boldsymbol{Z}^{-1}[Y(z)] = 5[-\delta(n) + 2\mathrm{u}(n)] = -5\delta(n) + 10\mathrm{u}(n)\end{aligned}$$

Thus the step response y(n)=-5δ(n)+10u(n)

## Problem 6.5

Given the following difference equation

$$y(n) = x(n) - 0.5y(n-1)$$

1. Find H(z).
2. Determine the impulse response y(n) if the input x(n)=δ(n).

3. Determine the step response y(n) if the input x(n)=u(n).

## solution

1. Find H(z).

$$\begin{aligned}\sum_{n=0}^{\infty}[y(n)+0.5y(n-1)]z^{-n} &= \sum_{n=0}^{\infty}x(n)z^{-n}\\ Y(z)+0.5z^{-1}[y(-1)z+Y(z)] &= X(z) \quad \{y(-1)=0\}\\ [1+0.5z^{-1}]Y(z) &= X(z)\\ H(z)\equiv\frac{Y(z)}{X(z)} &= \frac{1}{1+0.5z^{-1}}\end{aligned}$$

Thus $H(z)=\frac{1}{1+0.5z^{-1}}$

2. x(n)=δ(n), find y(n)

$$\begin{aligned}X(z) &= \sum_{n=0}^{\infty}\delta(n)z^{-n}=1\\ Y(z) &= H(z)\cdot X(z)=\frac{1}{1+0.5z^{-1}}\cdot 1=\frac{1}{1+0.5z^{-1}}\\ y(n) &= \boldsymbol{Z}^{-1}[Y(z)]=(-0.5)^n\mathrm{u}(n)\end{aligned}$$

Thus the impulse response y(n)=$(-0.5)^n\mathrm{u}(n)$

3. x(n)=u(n), find y(n)

$$\begin{aligned}X(z) &= \sum_{n=0}^{\infty}\mathrm{u}(n)z^{-n}=\frac{1}{1-z^{-1}}\\ Y(z) &= H(z)\cdot X(z)=\frac{1}{1+0.5z^{-1}}\cdot\frac{1}{1-z^{-1}}=\frac{z^2}{(z+0.5)(z-1)}\\ \frac{Y(z)}{z} &= \frac{z}{(z+0.5)(z-1)}=\frac{\frac{z}{z-1}\big|_{z=-0.5}}{z+0.5}+\frac{\frac{z}{z+0.5}\big|_{z=1}}{z-1}=\frac{\frac{1}{3}}{z+0.5}+\frac{\frac{2}{3}}{z-1}\\ Y(z) &= \frac{\frac{1}{3}}{1+0.5z^{-1}}+\frac{\frac{2}{3}}{1-z^{-1}}\\ y(n) &= \boldsymbol{Z}^{-1}[Y(z)]=[\frac{1}{3}(-0.5)^n+\frac{2}{3}]\mathrm{u}(n)\end{aligned}$$

Thus the step response y(n)=$[\frac{1}{3}(-0.5)^n+\frac{2}{3}]\mathrm{u}(n)$

# Problem 6.6

A digital system is described by the following difference equation:

$$y(n)=x(n)-0.25x(n-2)-1.1y(n-1)-0.28y(n-2)$$

Find the transfer function H(z), the denominator polynomial A(z), and the numerator polynomial B(z).

## solution

$$y(n) + 1.1y(n-1) + 0.28y(n-2) = x(n) - 0.25x(n-2)$$
$$\sum_{n=0}^{\infty}[y(n) + 1.1y(n-1) + 0.28y(n-2)]z^{-n} = \sum_{n=0}^{\infty}[x(n) - 0.25x(n-2)]z^{-n}$$
$$Y(z) + 1.1z^{-1}[zy(-1) + Y(z)] + 0.28z^{-2}[z^2y(-2) + zy(-1) + Y(z)]$$
$$= X(z) - 0.25z^{-2}[z^2x(-2) + zx(-1) + X(z)] \quad \{x(-2) = x(-1) = y(-2) = y(-1) = 0\}$$
$$[1 + 1.1z^{-1} + 0.28z^{-2}]Y(z) = [1 - 0.25z^{-2}]X(z)$$
$$H(z) \equiv \frac{B(z)}{A(z)} = \frac{Y(z)}{X(z)} = \frac{1 - 0.25z^{-2}}{1 + 1.1z^{-1} + 0.28z^{-2}}$$

Thus H(z) = $\frac{1-0.25z^{-2}}{1+1.1z^{-1}+0.28z^{-2}}$, the denominator A(z)=$1 + 1.1z^{-1} + 0.28z^{-2}$, the numerator B(z)=$1 - 0.25z^{-2}$

# Problem 6.9

A digital system is described by the following difference equation:

$$y(n) = x(n) - 0.3x(n-1) + 0.28x(n-2)$$

Find the transfer function H(z), the denominator polynomial A(z), and the numerator polynomial B(z).

## solution

$$y(n) = x(n) - 0.3x(n-1) + 0.28x(n-2)$$
$$\sum_{n=0}^{\infty} y(n)z^{-n} = \sum_{n=0}^{\infty}[x(n) - 0.3x(n-1) + 0.28x(n-2)]z^{-n}$$
$$Y(z) = X(z) - 0.3z^{-1}[zx(-1) + X(z)] + 0.28z^{-2}[z^2x(-2) + zx(-1) + X(z)] \quad \{x(-2) = x(-1) = 0\}$$
$$Y(z) = [1 - 0.3z^{-1} + 0.28z^{-2}]X(z)$$
$$H(z) \equiv \frac{B(z)}{A(z)} = \frac{Y(z)}{X(z)} = \frac{1 - 0.3z^{-1} + 0.28z^{-2}}{1}$$

Thus H(z) = $1 - 0.3z^{-1} + 0.28z^{-2}$, the denominator A(z)=$1$, the numerator B(z)=$1 - 0.3z^{-1} + 0.28z^{-2}$

# Problem 6.11

Convert each of the following transfer functions into the difference equations:

1. $H(z) = 0.1 + 0.2z^{-1} + 0.3z^{-2}$
2. $H(z) = \frac{0.5-0.5z^{-2}}{1-0.3z^{-1}+0.8z^{-2}}$

## solution

1. Here $X(z) \equiv \sum_{n=0}^{\infty} x(n)z^{-n}, Y(z) \equiv \sum_{n=0}^{\infty} y(n)z^{-n}, H(z) = \frac{B(z)}{A(z)}$

   Having

$$
\begin{aligned}
A(z)Y(z) &= B(z)X(z) \\
1 \cdot Y(z) &= [0.1 + 0.2z^{-1} + 0.3z^{-2}] \cdot X(z) \\
\sum_{n=0}^{\infty} y(n)z^{-n} &= 0.1\sum_{n=0}^{\infty} x(n)z^{-n} + 0.2\sum_{n=0}^{\infty} x(n)z^{-(n+1)} + 0.3\sum_{n=0}^{\infty} x(n)z^{-(n+2)} \\
\sum_{n=0}^{\infty} y(n)z^{-n} &= 0.1\sum_{n=0}^{\infty} x(n)z^{-n} + 0.2\sum_{n=1}^{\infty} x(n-1)z^{-n} + 0.3\sum_{n=2}^{\infty} x(n-2)z^{-n} \\
\sum_{n=0}^{\infty} y(n)z^{-n} &= \sum_{n=0}^{\infty} \{0.1x(n) + 0.2x(n-1)z^{-n} + 0.3x(n-2)\}\, z^{-n} \\
&\quad - 0.2x(-1) - 0.3[x(-2) + x(-1)z^{-1}] \quad \{x(-2) = x(-1) = 0\} \\
\sum_{n=0}^{\infty} y(n)z^{-n} &= \sum_{n=0}^{\infty} \{0.1x(n) + 0.2x(n-1)z^{-n} + 0.3x(n-2)\}\, z^{-n} \\
y(n) &= 0.1x(n) + 0.2x(n-1)z^{-n} + 0.3x(n-2) \quad (n \geq 0)
\end{aligned}
$$

Thus the difference equation $y(n) = 0.1x(n) + 0.2x(n-1)z^{-n} + 0.3x(n-2) \quad (n \geq 0)$

2. Here $X(z) \equiv \sum_{n=0}^{\infty} x(n)z^{-n}, Y(z) \equiv \sum_{n=0}^{\infty} y(n)z^{-n}, H(z) = \frac{B(z)}{A(z)}$

   Having

$$
\begin{aligned}
A(z)Y(z) &= B(z)X(z) \\
[1 - 0.3z^{-1} + 0.8z^{-2}] \cdot Y(z) &= [0.5 - 0.5z^{-2}] \cdot X(z) \\
\sum_{n=0}^{\infty} y(n)z^{-n} - 0.3\sum_{n=0}^{\infty} y(n)z^{-(n+1)} + 0.8\sum_{n=0}^{\infty} y(n)z^{-(n+2)} & \\
= 0.5\sum_{n=0}^{\infty} x(n)z^{-n} - 0.5\sum_{n=0}^{\infty} x(n)z^{-(n+2)} & \\
\sum_{n=0}^{\infty} y(n)z^{-n} - 0.3\sum_{n=1}^{\infty} y(n-1)z^{-n} + 0.8\sum_{n=2}^{\infty} y(n-2)z^{-n} & \\
= 0.5\sum_{n=0}^{\infty} x(n)z^{-n} - 0.5\sum_{n=2}^{\infty} x(n-2)z^{-n} &
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \{y(n) - 0.3y(n-1) + 0.8y(n-2)\}z^{-n} + 0.3y(-1) - 0.8[y(-2) + z^{-1}y(-1)] \\
&\quad = \sum_{n=0}^{\infty} \{0.5x(n) - 0.5x(n-2)\}z^{-n} + 0.5[x(-2) + z^{-1}x(-1)] \\
&\quad \{y(-2) = y(-1) = x(-2) = x(-1) = 0\} \\
&\sum_{n=0}^{\infty} \{y(n) - 0.3y(n-1) + 0.8y(n-2)\}z^{-n} = \sum_{n=0}^{\infty} \{0.5x(n) - 0.5x(n-2)\}z^{-n} \\
&\quad y(n) - 0.3y(n-1) + 0.8y(n-2) = 0.5x(n) - 0.5x(n-2) \quad (n \geq 0) \\
&\quad y(n) = 0.5x(n) - 0.5x(n-2) + 0.3y(n-1) - 0.8y(n-2) \quad (n \geq 0)
\end{aligned}
$$

Thus the difference equation $y(n) = 0.5x(n) - 0.5x(n-2) + 0.3y(n-1) - 0.8y(n-2) \quad (n \geq 0)$

## Problem 6.15

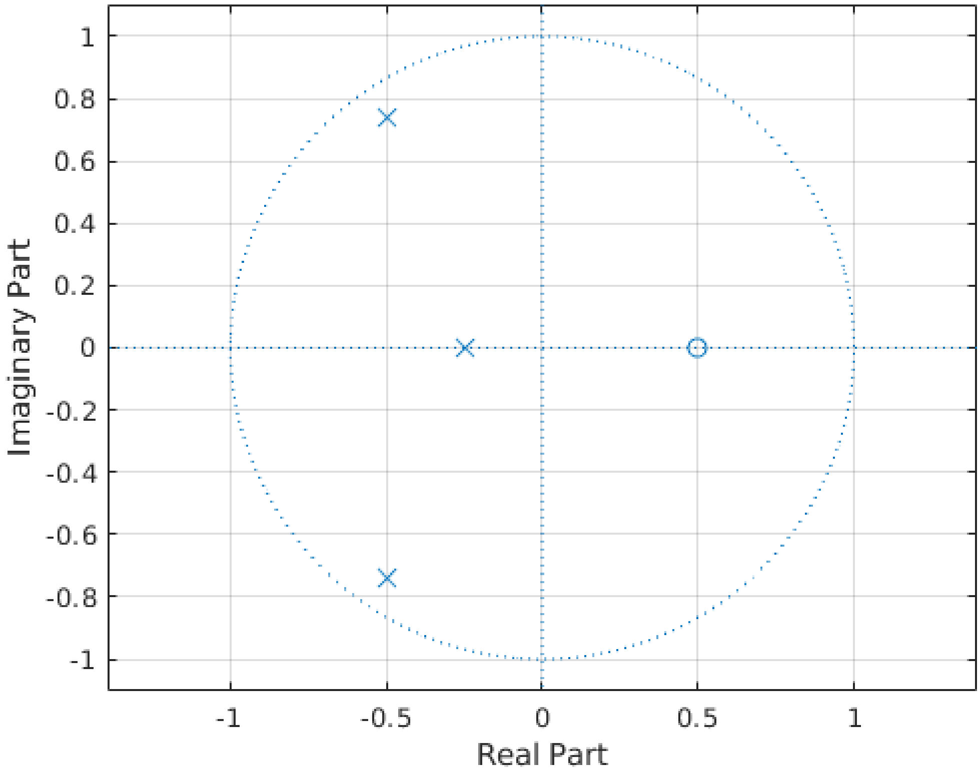
Given each of the following transfer functions that describe the digital systems, sketch the z-plane pole-zero plot and determine the stability for each digital system

1. $H(z) = \frac{z-0.5}{(z+0.25)(z^2+z+0.8)}$
2. $H(z) = \frac{z^2+0.25}{(z-0.5)(z^2+4z+7)}$
3. $H(z) = \frac{z+0.95}{(z+0.2)(z^2+1.414z+1)}$
4. $H(z) = \frac{z^2+z+0.25}{(z-1)(z+1)^2(z-0.36)}$

## solution

1. zero: z= 0.5, poles: z= -0.25 (|z|=0.25), z= -0.5±0.7416j (|z|=0.8944), stable.

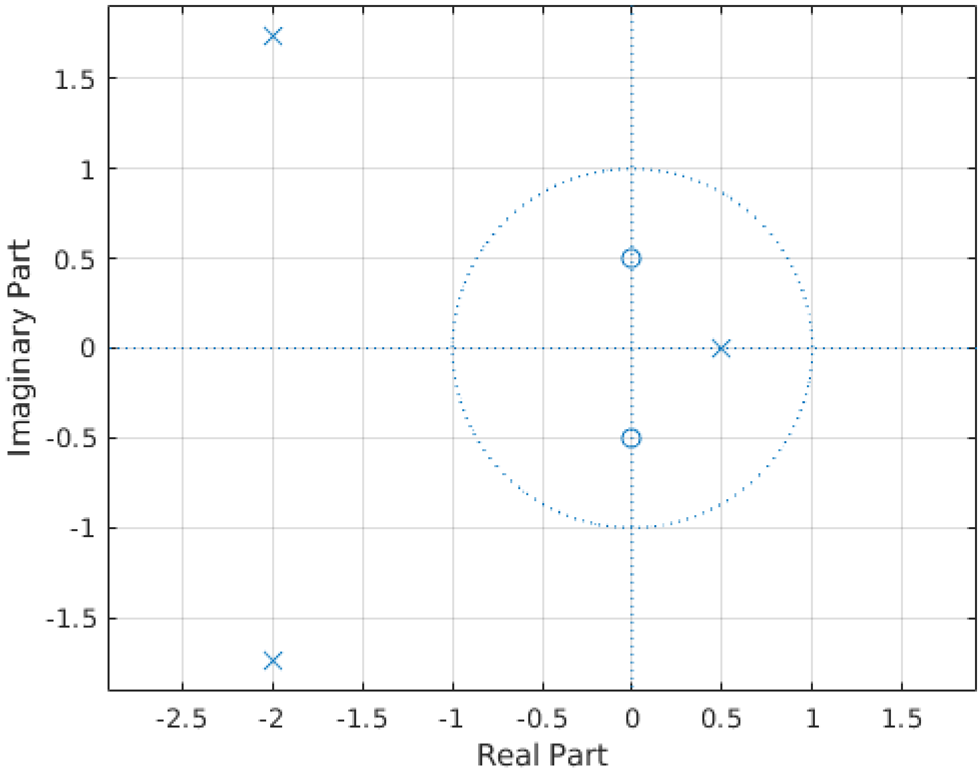
```
a = conv([1, 0.25], [1, 1, 0.8]); % coef of denominator(z^{-1})
b = [1, -0.5];
N = length(a) - 1;
M = length(b) - 1;
b = [zeros(1, N-M), b]; % coef of numerator(z^{-1})
figure; zplane(b,a); grid on % plot of zero-pole
```




2. zeros: z= ±0.5j, poles: z=0.5 (|z|=0.5), z= -2±1.7321j (|z|=2.6458), unstable.

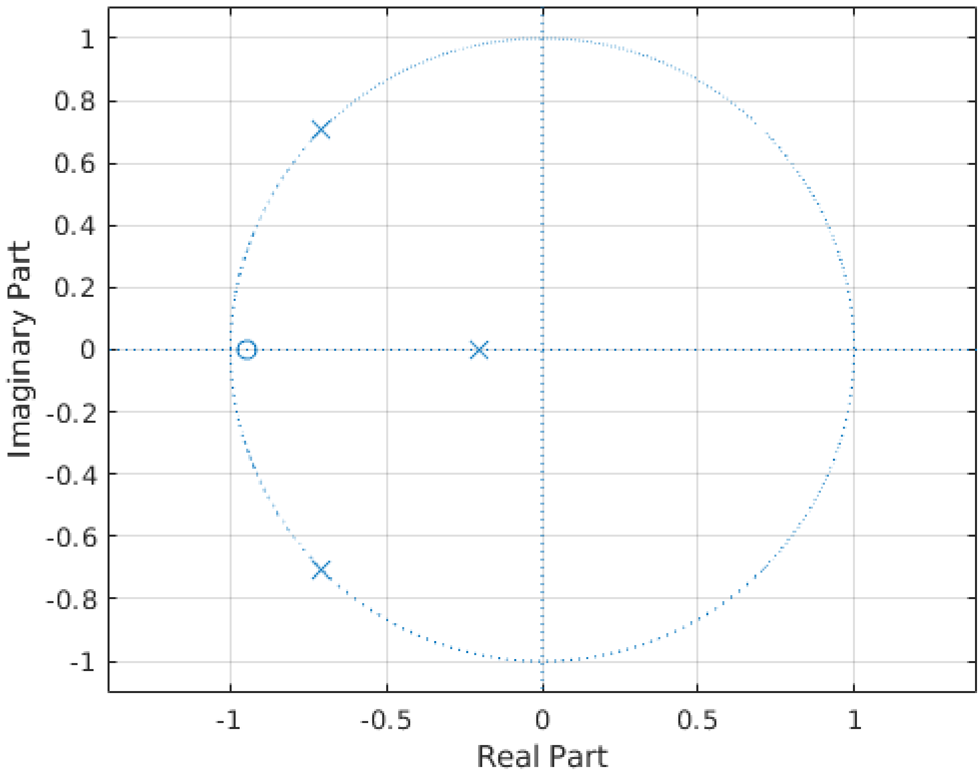
```
a = conv([1, -0.5], [1, 4, 7]); % coef of denominator(z^{-1})
b = [1, 0, 0.25];
N = length(a) - 1;
M = length(b) - 1;
b = [zeros(1, N-M), b]; % coef of numerator(z^{-1})

figure; zplane(b,a); grid on % plot of zero-pole
```



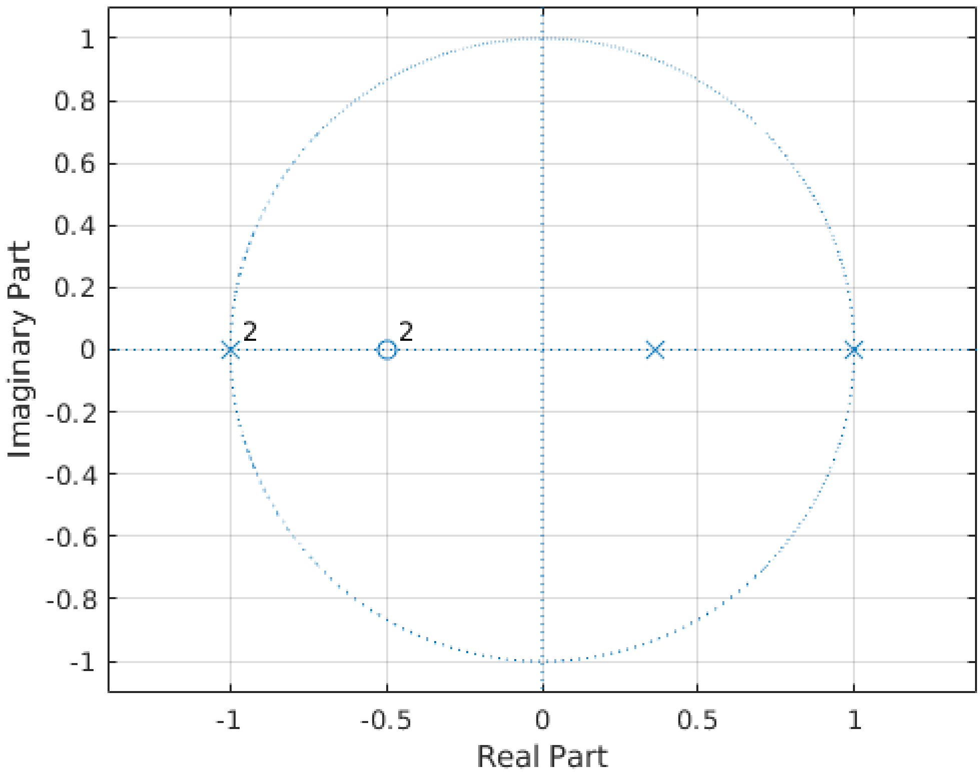

3. zero: z= -0.95, poles: z=0.2 (|z|=0.2), z= -0.7071±0.7071j (|z|=1), marginally stable.

```
a = conv([1, 0.2], [1, 1.414, 1]); % coef of denominator(z^{-1})
b = [1, 0.95];
N = length(a) - 1;
M = length(b) - 1;
b = [zeros(1, N-M), b]; % coef of numerator(z^{-1})
figure; zplane(b,a); grid on % plot of zero-pole
```




4. zeros: z= -0.5, z= -0.5, poles: z=1 (|z|=1), z= -1, z= -1 (|z|=1), z=0.36 (|z|=0.36), unstable.

```
a = conv(conv([1, -1], conv([1, 1], [1, 1])), [1, -0.36]); % coef of denominator(z^{-1})
b = [1, 1, 0.25];
N = length(a) - 1;
M = length(b) - 1;
b = [zeros(1, N-M), b]; % coef of numerator(z^{-1})
figure; zplane(b,a); grid on % plot of zero-pole
```




## Problem 6.16

Given the following digital system with a sampling rate of 8000 Hz

$$y(n) = 0.5x(n) + 0.5x(n-2)$$

1. Determine the frequency response.
2. Calculate and plot the magnitude and phase frequency responses.
3. Determine the filter type based on the magnitude frequency response.

### solution

1. Determine the frequency response $H(e^{j\Omega})$

   Do $\boldsymbol{Z}$ transform, x(-2)=x(-1)=0

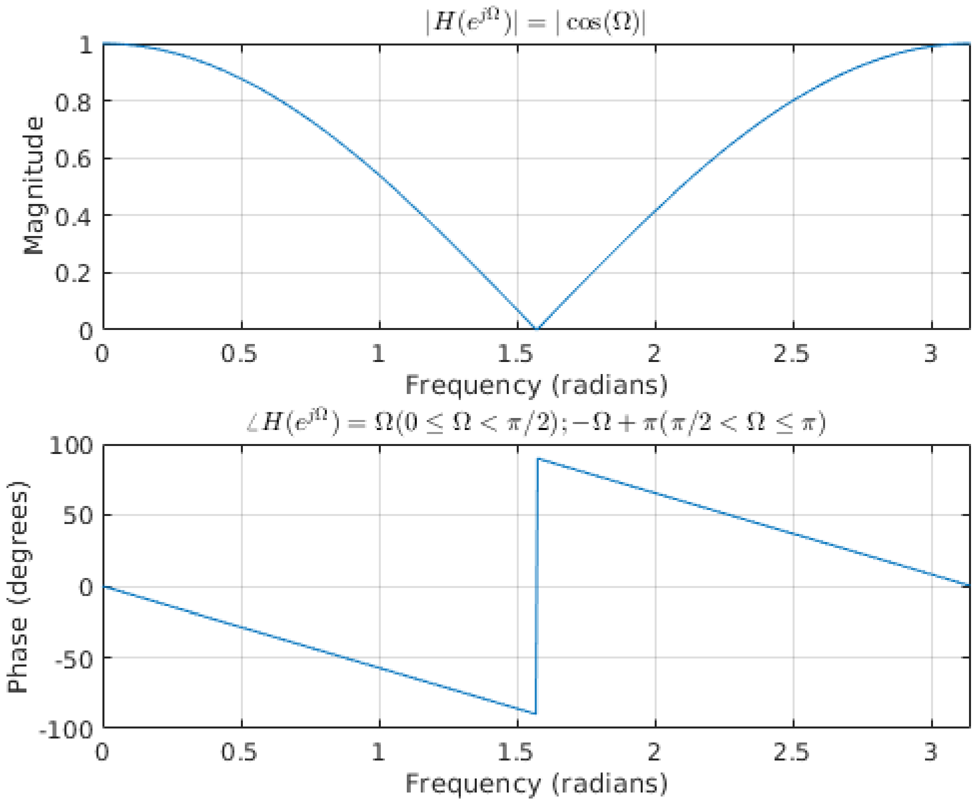
$$\begin{aligned} y(n) &= 0.5x(n) + 0.5x(n-2) \\ Y(z) &= 0.5X(z) + 0.5z^{-2}X(z) + 0.5[x(-2) + z^{-1}x(-1)] \\ &= [0.5 + 0.5z^{-2}]X(z) \\ H(z) &\equiv \frac{Y(z)}{X(z)} = 0.5 + 0.5z^{-2} \\ H(e^{j\Omega}) &= 0.5 + 0.5e^{-j2\Omega} = e^{-j\Omega}0.5[e^{j\Omega} + e^{-j\Omega}] \\ &= e^{-j\Omega}\cos(\Omega) \end{aligned}$$

   Thus $H(e^{j\Omega}) = e^{-j\Omega}\cos(\Omega)$

2. The magnitude $|H(e^{j\Omega})|$ and phase frequency responses $\angle H(e^{j\Omega})$

$$|H(e^{j\Omega})| = |\cos(\Omega)|$$
$$\angle H(e^{j\Omega}) = \begin{cases} -\Omega & 0 \leq \Omega < \pi/2 \\ -\Omega + \pi & \pi/2 < \Omega \leq \pi \end{cases}$$

```
clear; clc; close all
a = [1]; % coef of denominator(z^{-1})
b = [0.5, 0, 0.5]; % coef of numerator(z^{-1})
N = 1024; % 0~pi devided by 1024 points
[h, w] = freqz(b, a, N);
mag_h = abs(h); arg_h = 360/(2*pi) * angle(h);
figure();
subplot(2, 1, 1); plot(w, mag_h); xlim([0, pi]); grid on
xlabel('Frequency (radians)' ), ylabel('Magnitude' );
title('$|H(e^{j\Omega})| = |\cos(\Omega)|$' , 'Interpreter','LaTex');
subplot(2, 1, 2); plot(w, arg_h); xlim([0, pi]); grid on
xlabel('Frequency (radians)' ), ylabel('Phase (degrees)' )
title('$\angle H(e^{j\Omega})= \Omega (0\leq \Omega < \pi/2); -\Omega+\pi (\pi/2 < \Omega
\leq \pi)$', 'Interpreter','LaTex');
```




3. The filter type: **band stop filter**

## Problem 6.18

For the following digital system with a sampling rate of 8000 Hz,

$$y(n) = 0.5x(n) + 0.5y(n-1)$$

1. Determine the frequency response.

2. Calculate and plot the magnitude and phase frequency responses.
3. Determine the filter type based on the magnitude frequency response.

## solution

1. Determine the frequency response $H(e^{j\Omega})$

   Do $\boldsymbol{Z}$ transform, x(-2)=x(-1)=0

$$
\begin{aligned}
y(n) - 0.5y(n-1) &= 0.5x(n) \\
Y(z) - 0.5z^{-1}Y(z) - 0.5y(-1) &= 0.5X(z) \\
[1 - 0.5z^{-1}]Y(z) &= 0.5X(z) \\
H(z) \equiv \frac{Y(z)}{X(z)} &= \frac{0.5}{1-0.5z^{-1}} \\
H(e^{j\Omega}) &= \frac{0.5}{1-0.5e^{-j\Omega}} = \frac{0.5}{1-0.5[\cos(\Omega) - j\sin(\Omega)]} \\
&= \frac{0.5}{[1-0.5\cos(\Omega)] + j0.5\sin(\Omega)} \\
&= \frac{0.5 \times \{[1-0.5\cos(\Omega)] - j0.5\sin(\Omega)\}}{[1-0.5\cos(\Omega)]^2 + 0.5^2\sin^2(\Omega)} \\
&= \frac{0.5}{\sqrt{1.25-\cos(\Omega)}} \frac{\{[1-0.5\cos(\Omega)] - j0.5\sin(\Omega)\}}{\sqrt{[1-0.5\cos(\Omega)]^2 + 0.5^2\sin^2(\Omega)}} \\
&= \frac{0.5}{\sqrt{1.25-\cos(\Omega)}} e^{-\phi} \quad [\phi = \tan^{-1}(\frac{0.5\sin(\Omega)}{1-0.5\cos(\Omega)})]
\end{aligned}
$$

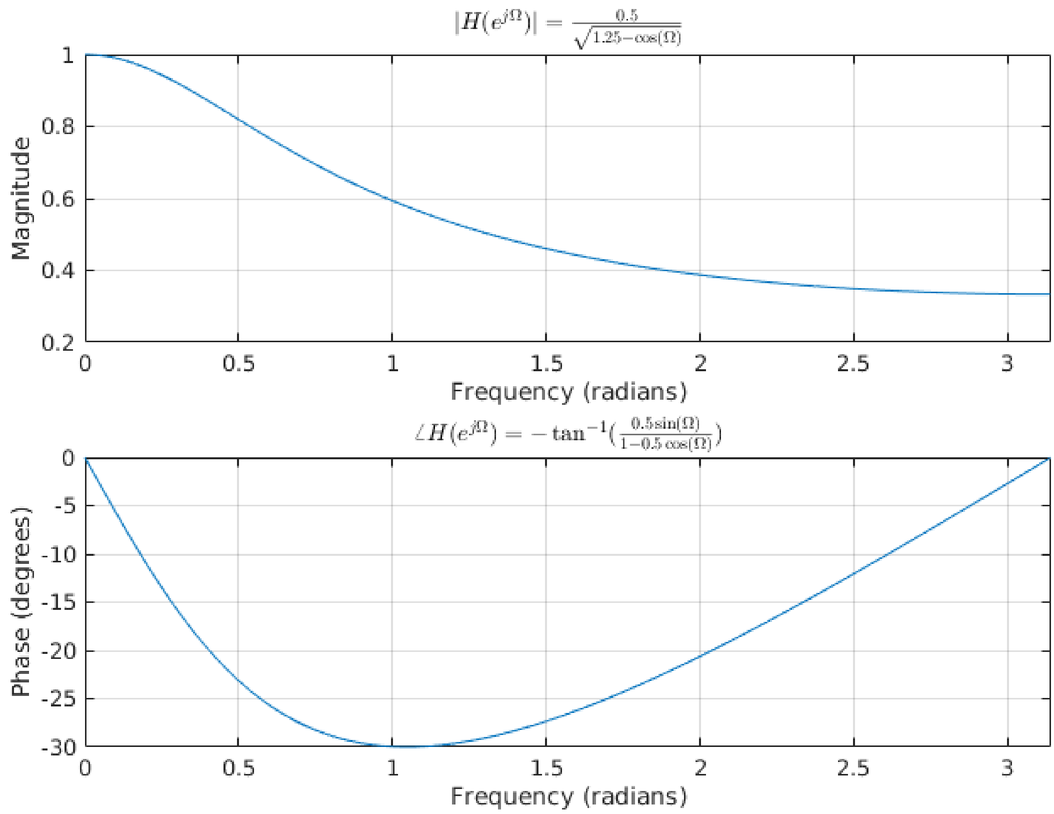
   Because $1 - 0.5\cos(\Omega) \geq 1 - 0.5 = 0.5 > 0$, and $\phi \in (-\pi/2, \pi/2)$

   Thus $H(e^{j\Omega}) = \frac{0.5}{\sqrt{1.25-\cos(\Omega)}} e^{-\phi} \quad [\phi = \tan^{-1}(\frac{0.5\sin(\Omega)}{1-0.5\cos(\Omega)})]$

2. The magnitude $|H(e^{j\Omega})|$ and phase frequency responses $\angle H(e^{j\Omega})$

$$
\begin{aligned}
|H(e^{j\Omega})| &= \frac{0.5}{\sqrt{1.25-\cos(\Omega)}} \\
\angle H(e^{j\Omega}) &= -\phi = -\tan^{-1}(\frac{0.5\sin(\Omega)}{1-0.5\cos(\Omega)})
\end{aligned}
$$

```
clear; clc; close all
a = [1, -0.5]; % coef of denominator(z^{-1})
b = [0.5]; % coef of numerator(z^{-1})
N = 1024; % 0~pi devided by 1024 points
[h, w] = freqz(b, a, N);
mag_h = abs(h); arg_h = 360/(2*pi) * angle(h);
figure();
subplot(2, 1, 1); plot(w, mag_h); xlim([0, pi]); grid on
xlabel('Frequency (radians)' ), ylabel('Magnitude' );
title('$|H(e^{j\Omega})| = \frac{0.5}{\sqrt{1.25-\cos(\Omega)}}$' ,
'Interpreter','LaTex');
subplot(2, 1, 2); plot(w, arg_h); xlim([0, pi]); grid on
xlabel('Frequency (radians)' ), ylabel('Phase (degrees)' )
title('$\angle H(e^{j\Omega})= -\tan^{-1}(\frac{0.5 \sin(\Omega)}{1-0.5\cos(\Omega)})$' ,
'Interpreter','LaTex');
```




3. The filter type: **low pass filter**

# Problem 6.25

Given the second-order IIR filter

$$H(z) = \frac{1 - 0.9z^{-1} - 0.1z^{-2}}{1 + 0.3z^{-1} - 0.04z^{-2}}$$

realize H(z) and develop difference equations using the following forms:

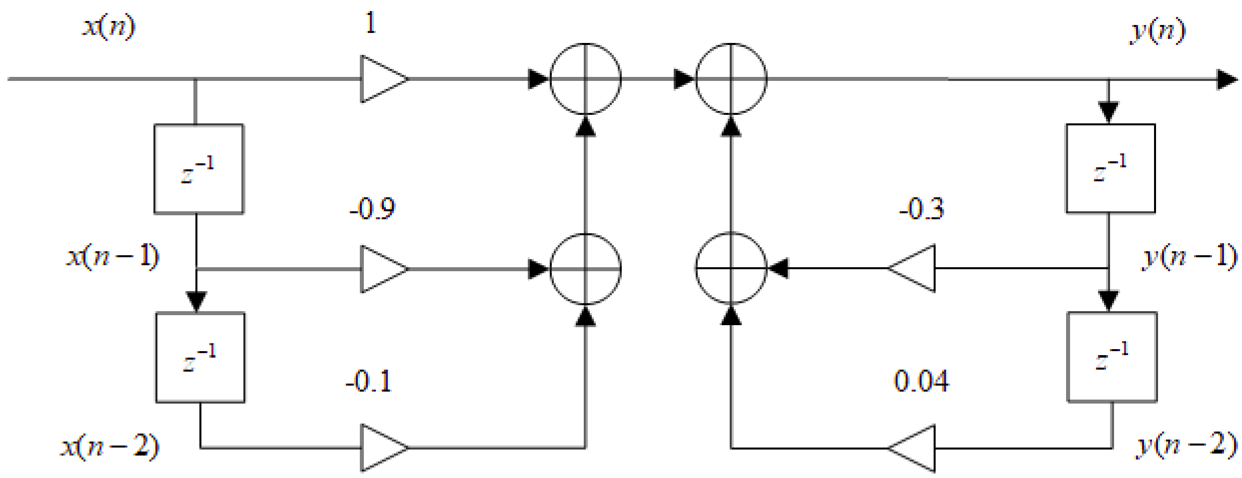
1. direct-form I
2. direct-form II

## solution

1. direct-form I

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \frac{1 - 0.9z^{-1} - 0.1z^{-2}}{1 + 0.3z^{-1} - 0.04z^{-2}}$$
$$A(z)Y(z) = B(z)X(z)$$
$$[1 + 0.3z^{-1} - 0.04z^{-2}]Y(z) = [1 - 0.9z^{-1} - 0.1z^{-2}]X(z)$$
$$y(n) + 0.3y(n-1) - 0.04y(n-2) = x(n) - 0.9x(n-1) - 0.1x(n-2)$$
$$y(n) = x(n) - 0.9x(n-1) - 0.1x(n-2) - 0.3y(n-1) + 0.04y(n-2)$$

The difference equation $y(n) = x(n) - 0.9x(n-1) - 0.1x(n-2) - 0.3y(n-1) + 0.04y(n-2)$




2. direct-form II

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \frac{1-0.9z^{-1}-0.1z^{-2}}{1+0.3z^{-1}-0.04z^{-2}}$$

$$W(z) \equiv \frac{1}{A(z)}X(z) \Rightarrow A(z)W(z) = X(z)$$

$$Y(z) = \frac{B(z)}{A(z)}X(z) = B(z)W(z)$$

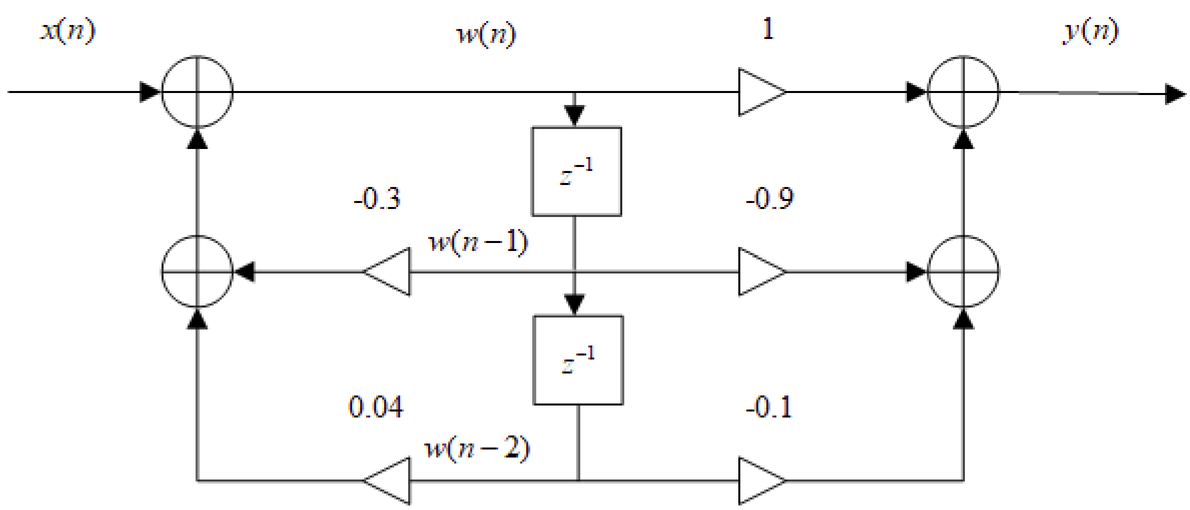
$$\begin{cases} X(z) = A(z)W(z) = [1+0.3z^{-1}-0.04z^{-2}]W(z) \\ Y(z) = B(z)W(z) = [1-0.9z^{-1}-0.1z^{-2}]W(z) \end{cases}$$

$$\begin{cases} x(n) = w(n) + 0.3w(n-1) - 0.04w(n-2) \\ y(n) = w(n) - 0.9w(n-1) - 0.1w(n-2) \end{cases}$$

$$\begin{cases} w(n) = x(n) - 0.3w(n-1) + 0.04w(n-2) \\ y(n) = w(n) - 0.9w(n-1) - 0.1w(n-2) \end{cases}$$

The difference equation

$$\begin{cases} w(n) = x(n) - 0.3w(n-1) + 0.04w(n-2) \\ y(n) = w(n) - 0.9w(n-1) - 0.1w(n-2) \end{cases}$$




## Advanced Problems

### Problem 6.33

Let x(n)=$\{a_N, a_{N-1}, \cdots, a_0, a_1, \cdots, a_{N-1}, a_N\}$ be a finite-duration sequence, which is real and even. Show that if z=r$e^{j\theta}$ is a zero of X(z), then z=(1/r)$e^{-j\theta}$ is also a zero.

### solution

$$X(z) \equiv z^{-N}[\sum_{n=1}^{N} a_n z^n + a_0 + \sum_{n=1}^{N} a_n z^{-n}]$$

$$X(z)|_{z=re^{j\theta}} = r^{-N} e^{-j\theta N}[\sum_{n=1}^{N} a_n r^n e^{j\theta n} + a_0 + \sum_{n=1}^{N} a_n r^{-n} e^{-j\theta n}] = 0$$

$$\Rightarrow \sum_{n=1}^{N} a_n r^{-n} e^{-j\theta n} + a_0 + \sum_{n=1}^{N} a_n r^n e^{j\theta n} = 0$$

$$X(z)|_{z=r^{-1}e^{-j\theta}} = r^N e^{j\theta N}[\sum_{n=1}^{N} a_n r^{-n} e^{-j\theta n} + a_0 + \sum_{n=1}^{N} a_n r^n e^{j\theta n}] = 0$$

Thus z=(1/r)$e^{-j\theta}$ is also a zero.

## Problem 6.38

Let z=r$e^{j\Omega}$ be a zero inside of the unit circle, where 0<r<1, Consider a system whose transfer function as

$$H(z) = \frac{1}{1 - re^{j\theta} z^{-1}}$$

1. Show that the magnitude response is

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{1}{\sqrt{1 - 2r\cos(\Omega - \theta) + r^2}}$$

2. Show that the phase response is

$$\angle H\left(e^{j\Omega}\right) = -\tan^{-1}\left(\frac{r\sin(\Omega - \theta)}{1 - r\cos(\Omega - \theta)}\right)$$

3. Plot the magnitude response for r=0.8, θ=$60^\circ$, and Ω=0, π/4, π/2, 3π/4, π, respectively.

### solution

1. The magnitude response $\left|H\left(e^{j\Omega}\right)\right|$

$$
\begin{aligned}
H\left(e^{j\Omega}\right) &= \frac{1}{1-re^{j\theta}e^{-j\Omega}} \\
&= \frac{1}{1-r[\cos(\Omega-\theta)-j\sin(\Omega-\theta)]} \\
&= \frac{1}{[1-r\cos(\Omega-\theta)]+jr\sin(\Omega-\theta)} \\
&= \frac{[1-r\cos(\Omega-\theta)]-jr\sin(\Omega-\theta)}{[1-r\cos(\Omega-\theta)]^2+r^2\sin^2(\Omega-\theta)} \\
&= \frac{1}{\sqrt{[1-r\cos(\Omega-\theta)]^2+r^2\sin^2(\Omega-\theta)}}\frac{[1-r\cos(\Omega-\theta)]-jr\sin(\Omega-\theta)}{\sqrt{[1-r\cos(\Omega-\theta)]^2+r^2\sin^2(\Omega-\theta)}} \\
&= \frac{1}{\sqrt{1-2r\cos(\Omega-\theta)+r^2}}e^{-j\phi} \quad \left[\phi=\tan^{-1}\left(\frac{r\sin(\Omega-\theta)}{1-r\cos(\Omega-\theta)}\right)\right]
\end{aligned}
$$

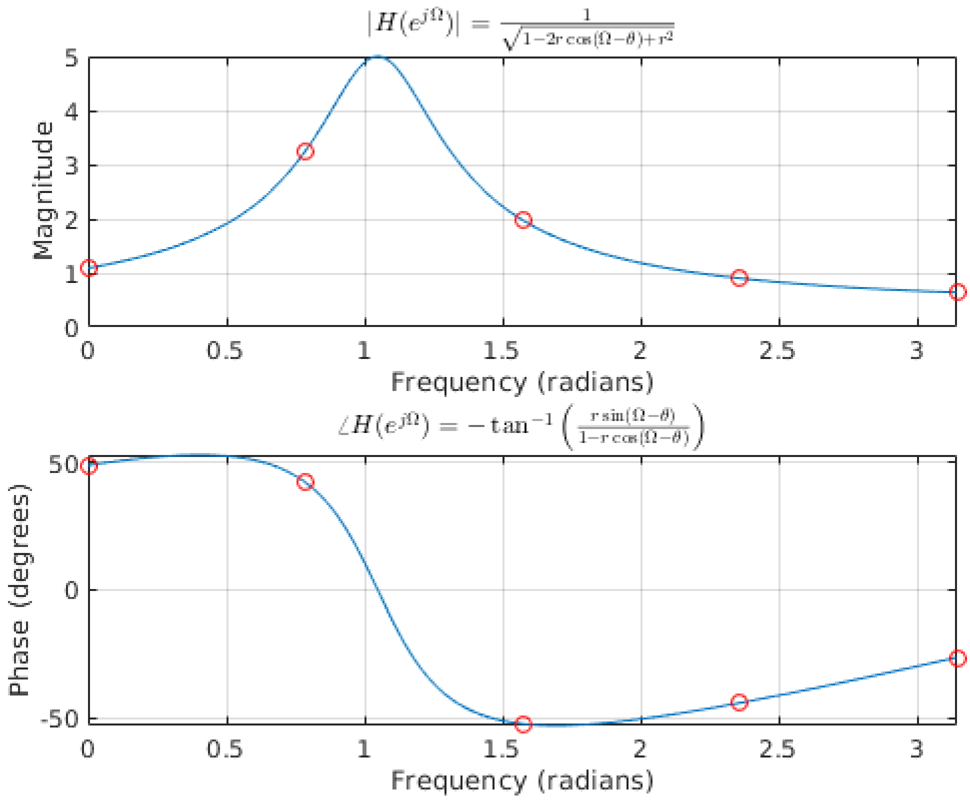
That is because $1-r\cos(\Omega-\theta)\geq 1-r>0$, and $\phi\in(-\pi/2,\pi/2)$

Thus $\left|H\left(e^{j\Omega}\right)\right| = \frac{1}{\sqrt{1-2r\cos(\Omega-\theta)+r^2}}$

2. Thus $\angle H\left(e^{j\Omega}\right) = -\phi = -\tan^{-1}\left(\frac{r\sin(\Omega-\theta)}{1-r\cos(\Omega-\theta)}\right)$

3. Plot the magnitude response for r=0.8, θ=$60^\circ$, and Ω=0, π/4, π/2, 3π/4, π, respectively.

```
clear; clc; close all
r = 0.8; theta = (2*pi)/360 * 60;
z_pole = r * exp(1i * theta);
a = [1, -z_pole]; % coef of denominator(z^{-1})
b = [1]; % coef of numerator(z^{-1})
N = 1024; % 0~pi devided by 1024 points
[h, w] = freqz(b, a, N);
mag_h = abs(h); arg_h = 360/(2*pi) * angle(h);
list_index = 1 + floor([0, 1/4, 1/2, 3/4] * N); % find index for 0, pi/4, pi/2, 3*pi/4
w_special = w(list_index); mag_special = mag_h(list_index); arg_special =
arg_h(list_index); % add index for pi
w_special = [w_special; pi]; mag_special = [mag_special; mag_h(end)]; arg_special =
[arg_special; arg_h(end)];
figure();
subplot(2, 1, 1); plot(w, mag_h); xlim([0, pi]); grid on;
hold on; plot(w_special, mag_special, 'ro');
xlabel('Frequency (radians)'), ylabel('Magnitude');
title('$|H(e^{j\Omega})|=\frac{1}{\sqrt{1-2 r \cos (\Omega-\theta)+r^{2}}}$' ,
'Interpreter','LaTex');
subplot(2, 1, 2); plot(w, arg_h); xlim([0, pi]); grid on
hold on; plot(w_special, arg_special, 'ro');
xlabel('Frequency (radians)'), ylabel('Phase (degrees)')
title('$\angle H(e^{j\Omega})= -\tan ^{-1}\left(\frac{r \sin (\Omega-\theta)}{1-r \cos
(\Omega-\theta)}\right)$' , 'Interpreter','LaTex');
```




# MATLAB Projects

## Problem 6.29

Given a filter

$$H(z) = \frac{1 - z^{-1} + z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

1. Plot the magnitude frequency response and phase response using MATLAB.
2. Specify the type of filtering.
3. Find the difference equation.
4. Repeat (d) using the MATLAB function filter().

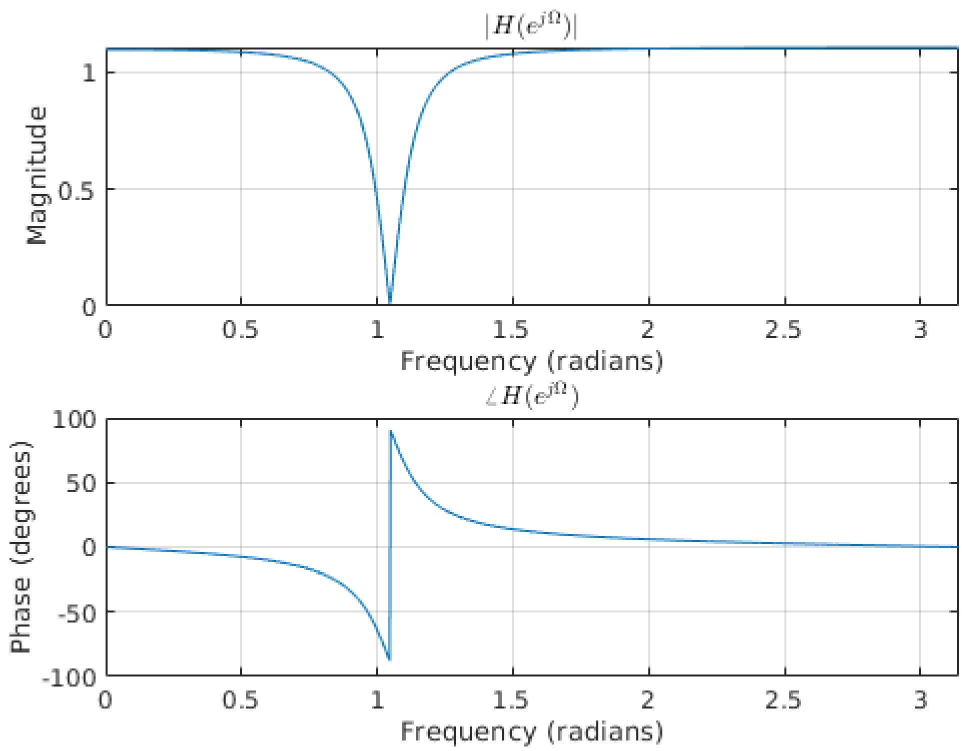
   [ Perform filtering, that is, calculate y(n) for first 1000 samples for each of the following inputs and plot the filter outputs using MATLAB, assuming that all initial conditions are zeros and the sampling rate is 8000 Hz: ]

$$x(n) = \cos\left(\pi \cdot 10^3 \frac{n}{8{,}000}\right)$$
$$x(n) = \cos\left(\frac{8}{3}\pi \cdot 10^3 \frac{n}{8{,}000}\right)$$
$$x(n) = \cos\left(6\pi \cdot 10^3 \frac{n}{8{,}000}\right)$$

**solution**

1. Plot the magnitude frequency response and phase response using MATLAB.

```
clear; clc; close all
a = [1, -0.9, 0.81]; % coef of denominator(z^{-1})
b = [1, -1, 1]; % coef of numerator(z^{-1})
N = 1024; % 0~pi devided by 1024 points
[h, w] = freqz(b, a, N);
mag_h = abs(h); arg_h = 360/(2*pi) * angle(h);
figure();
subplot(2, 1, 1); plot(w, mag_h); xlim([0, pi]); grid on
xlabel('Frequency (radians)' ), ylabel('Magnitude' );
title('$|H(e^{j\Omega})|$' , 'Interpreter' ,'LaTex' );
subplot(2, 1, 2); plot(w, arg_h); xlim([0, pi]); grid on
xlabel('Frequency (radians)' ), ylabel('Phase (degrees)' )
title('$\angle H(e^{j\Omega})$' , 'Interpreter' ,'LaTex' );
```




2. The filter type: **band stop filter**

3. Find the difference equation.

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \frac{1 - z^{-1} + z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

$$A(z)Y(z) = [1 - 0.9z^{-1} + 0.81z^{-2}]Y(z)$$

$$= B(z)X(z) = [1 - z^{-1} + z^{-2}]X(z)$$

$$y(n) - 0.9y(n-1) + 0.81y(n-2) = x(n) - x(n-1) + x(n-2)$$

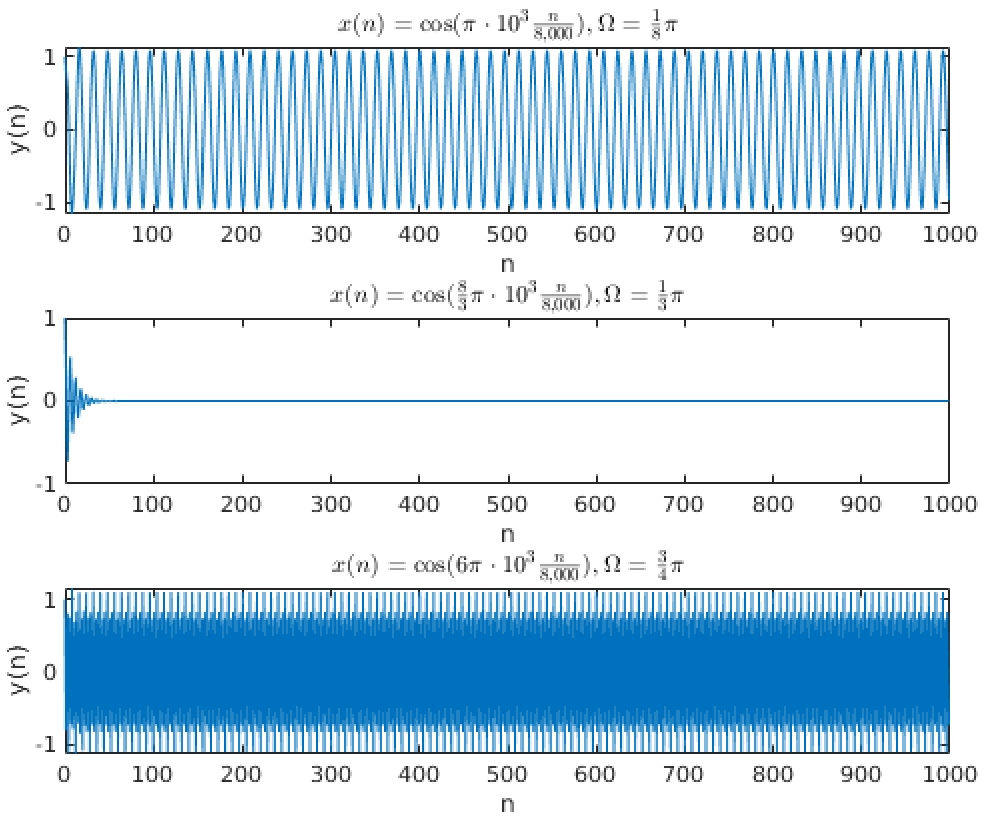
$$y(n) = x(n) - x(n-1) + x(n-2) + 0.9y(n-1) - 0.81y(n-2)$$

Thus the difference equation:

$$y(n) = x(n) - x(n-1) + x(n-2) + 0.9y(n-1) - 0.81y(n-2)$$

4. calculate y(n) for first 1000 samples

```
clear; clc; close all
%% input series
fs = 8000;
func1 = @(n) cos(pi * 1000 * n / fs);
func2 = @(n) cos((8/3)*pi * 1000 * n / fs);
func3 = @(n) cos(6*pi * 1000 * n / fs);
list_n = 0:1:1000-1; % first 1000 samples, 0~999
x1 = func1(list_n);
x2 = func2(list_n);
x3 = func3(list_n);
%% output series after filtering
a = [1, -0.9, 0.81]; % coef of denominator(z^{-1})
b = [1, -1, 1]; % coef of numerator(z^{-1})
figure();
y1 = filter(b, a, x1); subplot(3, 1, 1); plot(list_n, y1)
xlabel('n'); ylabel('y(n)'); title('$x(n)=\cos (\pi \cdot 10^{3} \frac{n}{8,000}), \Omega
= \frac{1}{8}\pi$' , 'Interpreter','LaTex');
y2 = filter(b, a, x2); subplot(3, 1, 2); plot(list_n, y2)
xlabel('n'); ylabel('y(n)'); title('$x(n)=\cos (\frac{8}{3}\pi \cdot 10^{3} \frac{n}
{8,000}), \Omega = \frac{1}{3}\pi$' , 'Interpreter','LaTex');
y3 = filter(b, a, x3); subplot(3, 1, 3); plot(list_n, y3)
xlabel('n'); ylabel('y(n)'); title('$x(n)=\cos (6\pi \cdot 10^{3} \frac{n}{8,000}),
\Omega = \frac{3}{4}\pi$' , 'Interpreter','LaTex');
```




**conclusion**:

After the **band stop** filter, the component at $\Omega = \frac{1}{3}\pi$ is filtered;

the components at $\Omega = \frac{1}{8}\pi, \frac{3}{4}\pi$ remain.

## Problem 6.32

Echo generation (sound regeneration):

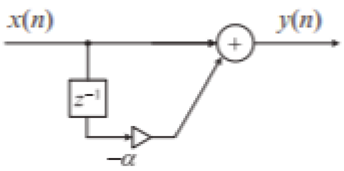



Fig 6.36 A preemphasis filter.

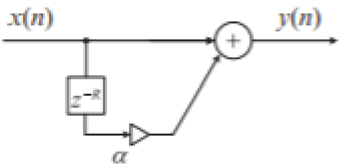



Fig 6.37 A single echo generator using an FIR filter.

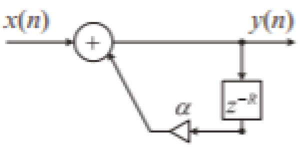



Fig 6.38 A multiple-echo generator using an IIR filter.

Echo is the repetition of sound due to sound wave reflection from the objects. It can easily be generated using an FIR filter shown in Fig. 6.37:where |α|<1 is an attenuation factor and R is the delay of the echo. However, a single echo generator may not be useful, so a multiple-echo generator using an IIR filter is usually applied, as shown in Fig. 6.38. As shown in Fig. 6.38, an echo signal is generated by the sum of delayed versions of sound with attenuation and the non-delayed version given by

$$y(n) = x(n) + \alpha x(n-R) + \alpha^2 x(n-2R) + \cdots = \sum_{k=0}^{\infty} \alpha^k x(n-kR)$$

where α is the attenuation factor. Applying z-transform, it follows that.

$$Y(z) = X(z)\sum_{k=0}^{\infty}\left(\alpha z^{-R}\right)^k = X(z)\frac{1}{1-\alpha z^{-R}} \text{ for } \left|\alpha z^{-R}\right| < 1$$

Thus, we yield the transfer function and difference equation below:

$$H(z) = \frac{1}{1-\alpha z^{-R}}$$

and y(n)=x(n) +αy(n-1)

1. Assuming that the system has a sampling rate of 8000 Hz, plot the IIR filter frequency responses for the following cases: α=0.5 and R=1; α=0.6 and R=4; α=0.7 and

R=10, and characterize the frequency responses.

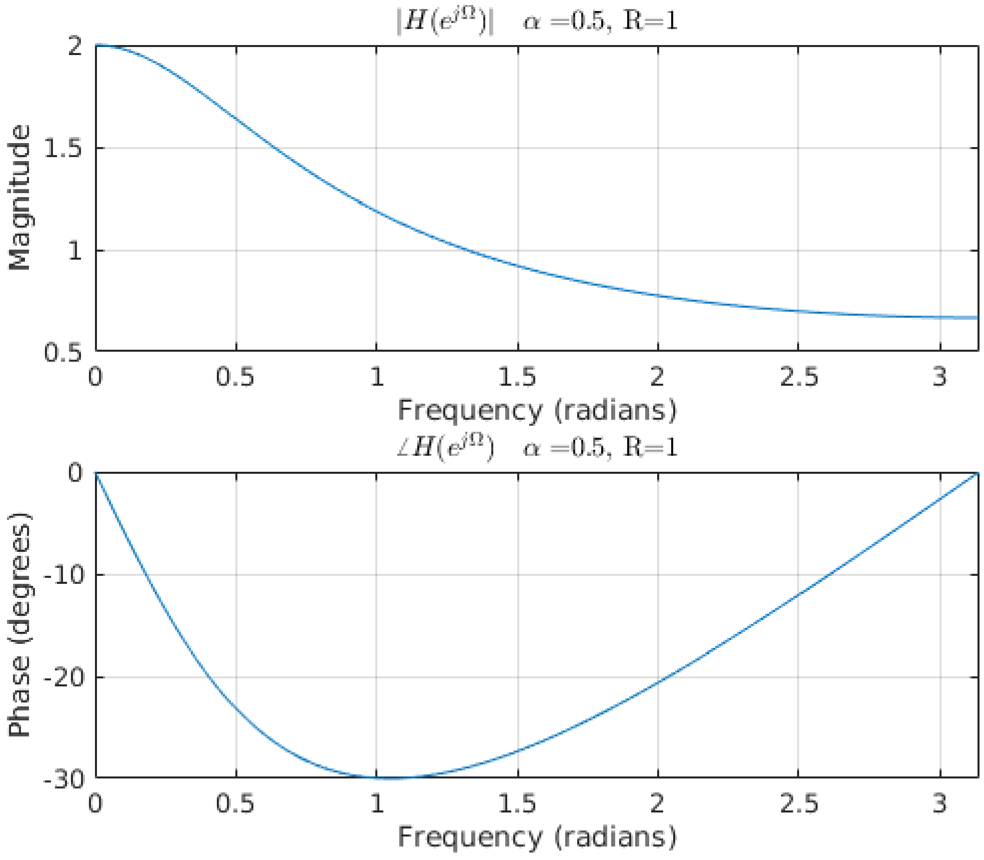
2. After implementing the multiple-echo generator using the following code:
   y = filter( [1] , [1 zeros(1, R-1) -alpha] , x )
   Evaluate the sound effects of processing the speech file ("sppech.dat") for the following cases: α=0.5 and R=500 (62.5ms); α=0.7 and R=1,000 (125ms); α=0.5, R=2,000 (250ms), and α=0.5, R=4,000 (500ms).

## solution

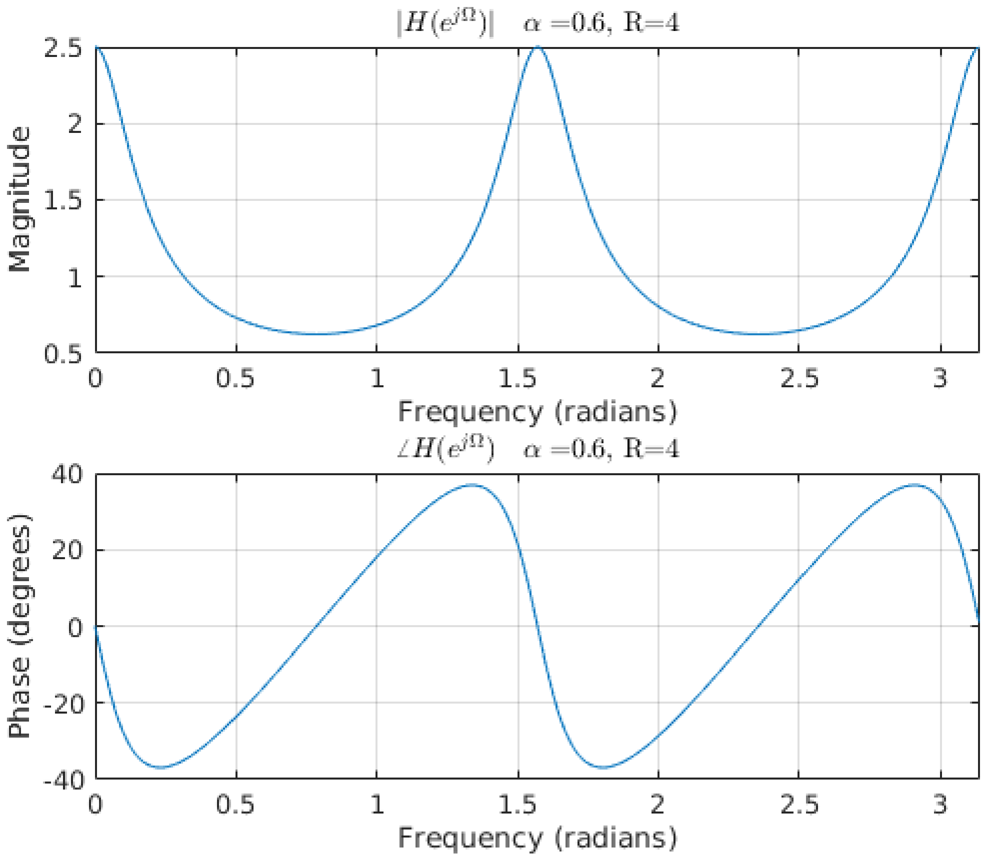
1. The frequency response:

```
function [] = freq_response_echo(alpha, R)
b = [1]; a = [1, zeros(1, R-1), -alpha]; N = 1024; % 0~pi divided by 1024 points
[h, w] = freqz(b, a, N);
mag_h = abs(h); arg_h = 360/(2*pi) * angle(h);
figure();
subplot(2, 1, 1); plot(w, mag_h); xlim([0, pi]); grid on
xlabel('Frequency (radians)' ), ylabel('Magnitude');
title(['$|H(e^{j\Omega})|\quad \alpha=$' , num2str(alpha), ', R=', num2str(R)],
'Interpreter','LaTex');
subplot(2, 1, 2); plot(w, arg_h); xlim([0, pi]); grid on
xlabel('Frequency (radians)' ), ylabel('Phase (degrees)' )
title(['$\angle H(e^{j\Omega})\quad \alpha=$' , num2str(alpha), ', R=', num2str(R)],
'Interpreter','LaTex');
end
```

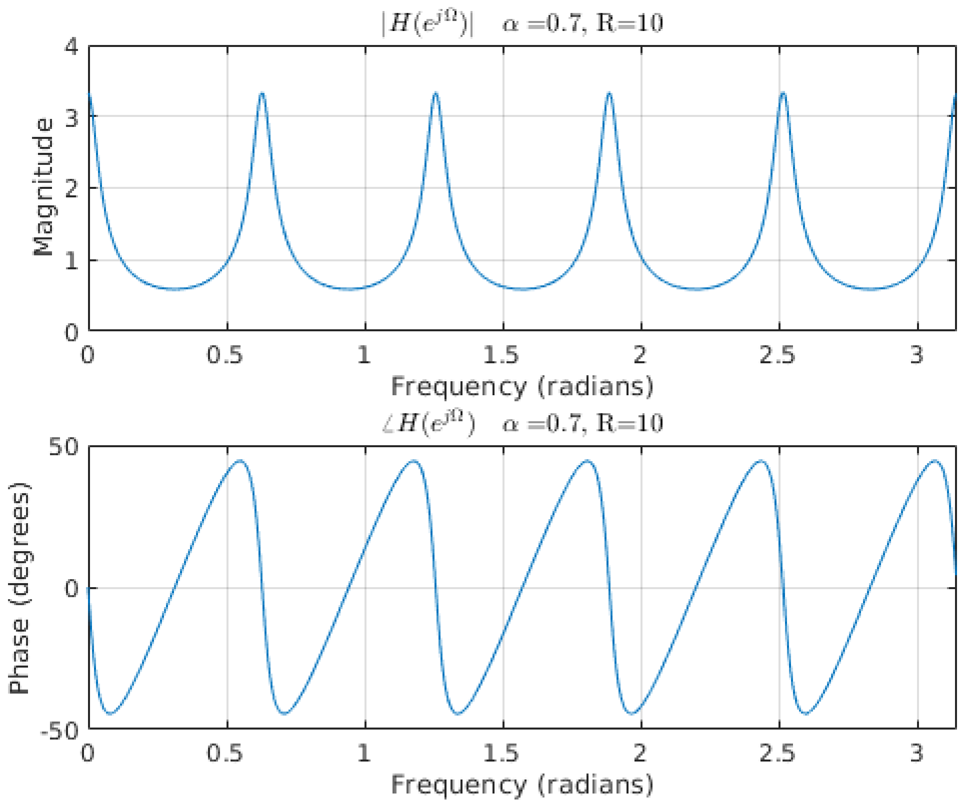
α=0.5 and R=1



α=0.6 and R=4




α=0.7 and R=10




2. Evaluate the sound effects of processing the speech file ("sppech.dat") for the following cases:

α=0.5 and R=500 (62.5ms);

α=0.7 and R=1,000 (125ms);

α=0.5, R=2,000 (250ms),

and α=0.5, R=4,000 (500ms).

Define **filter_echo.m** function

```
function y = filter_echo(alpha, R)
x = load('sppech.dat');
b = [1]; a = [1, zeros(1, R-1), -alpha];
y = filter(b, a, x); fs = 8000; y_max = max(max(y), min(y));
sound(y/y_max, fs);
end
```

Then apply the function to different cases:

For the "sppech.dat": "We lose the golden chain."

α=0.5 and R=500 (62.5ms);

We cannot hear the echo effect, hear the whole sentence "We lose the golden chain." clearly.

α=0.7 and R=1,000 (125ms);

We can clearly hear the echo effect.

α=0.5, R=2,000 (250ms),

We can also clearly hear the echo effect.

α=0.5, R=4,000 (500ms):

We can hear the same words "We lose" for 4 times