Homework 4

Course Title: Digital Signal Processing I (Spring 2020)

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Problems 5.1, 5.2, 5.4, 5.5, 5.9, 5.12, 5.14, 5.19, 5.20 (b), 5.21 (b)

Advanced problems

5.22, 5.23, 5.30

Problems

Problem 5.1

Find the z-transform for each of the following sequences:

(a) $x(n) = 4u(n)$ (b) $x(n) = (-0.7)^n u(n)$ (c) $x(n) = 4e^{-2n}u(n)$ (d) $x(n) = 4(0.8)^n \cos(0.1\pi n)u(n)$ (e) $x(n) = 4e^{-3n}\sin(0.1\pi n)u(n)$

solution

(a) For $|z| > 1$, z-transform:

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} 4z^{-n} = \frac{4}{1-z^{-1}} = \frac{4z}{z-1}$$

(b) For $|z| > 0.7$, z-transform:

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} (-0.7z^{-1})^n = \frac{1}{1+0.7z^{-1}} = \frac{z}{z+0.7}$$

(c) For $|z| > e^{-2}$, z-transform:

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} 4(e^{-2}z^{-1})^n = \frac{4}{1-e^{-2}z^{-1}} = \frac{4z}{z-e^{-2}}$$

(d) For $|z| > 0.8$, z-transform:

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} 4(0.8)^n \cos(0.1\pi n)z^{-n} = 4\sum_{n=0}^{\infty} \frac{1}{2}[(0.8e^{j0.1\pi}z^{-1})^n + (0.8e^{-j0.1\pi}z^{-1})^n] \\ &= 4\sum_{n=0}^{\infty} \frac{1}{2}[\frac{1}{1-0.8e^{j0.1\pi}z^{-1}} + \frac{1}{1-0.8e^{-j0.1\pi}z^{-1}}] = 4\frac{1-0.8\cos(0.1\pi)z^{-1}}{1-2\times 0.8\cos(0.1\pi)z^{-1} + 0.8^2 z^{-2}} \\ &= \frac{4-3.0434z^{-1}}{1-1.5217z^{-1}+0.64z^{-2}} = \frac{(4z-3.0434)z}{z^2-1.5217z+0.64} \end{aligned}$$

(e) For $|z| > e^{-3}$, z-transform:

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} 4e^{-3n}\sin(0.1\pi n)z^{-n} = 4\sum_{n=0}^{\infty}\frac{1}{j2}[(e^{-3}e^{j0.1\pi}z^{-1})^n - (e^{-3}e^{-j0.1\pi}z^{-1})^n]$$
$$= 4\frac{1}{j2}[\frac{1}{1-e^{-3}e^{j0.1\pi}z^{-1}} - \frac{1}{1-e^{-3}e^{-j0.1\pi}z^{-1}}]$$
$$= 4[\frac{e^{-3}\sin(0.1\pi)z^{-1}}{1-2e^{-3}\cos(0.1\pi)z^{-1}+e^{-6}z^{-2}}] = \frac{0.06154z^{-1}}{1-0.09470z^{-1}+0.002479z^{-2}}$$
$$= \frac{0.06154z}{z^2-0.09470z+0.002479}$$

Problem 5.2

Using the properties of the z-transform, find the z-transform for each of the following sequences:

(a) $x(n) = u(n) + (0.5)^n u(n)$ (b) $x(n) = e^{-3(n-4)}\cos(0.1\pi(n-4))u(n-4),$ where $u(n-4) = 1$ for $n \geq 4$ while $u(n-4) = 0$ for $n < 4$

solution

(a) For $|z| > \max(1, 0.5) = 1$, z-transform:

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty}(z^{-1})^n + (0.5z^{-1})^n = \sum_{n=0}^{\infty}(z^{-1})^n + \sum_{n=0}^{\infty}(0.5z^{-1})^n$$
$$= \frac{1}{1-z^{-1}} + \frac{1}{1-0.5z^{-1}}$$

(b) For $|z| > e^{-3}$, z-transform:

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} e^{-3(n-4)}\cos(0.1\pi(n-4))u(n-4)z^{-n}$$
$$= z^{-4}\sum_{n=4}^{\infty} e^{-3(n-4)}\cos(0.1\pi(n-4))z^{-(n-4)}$$
$$= z^{-4}\sum_{n=0}^{\infty} e^{-3n}\cos(0.1\pi n)z^{-n} = z^{-4}\frac{1}{2}[\frac{1}{1-e^{-3}e^{j0.1\pi}z^{-1}} + \frac{1}{1-e^{-3}e^{-j0.1\pi}z^{-1}}]$$
$$= z^{-4}\frac{1-e^{-3}\cos(0.1\pi)z^{-1}}{1-2e^{-3}\cos(0.1\pi)z^{-1}+e^{-6}z^{-2}}$$
$$= \frac{z^{-2}-0.04735z^{-3}}{z^2-0.09470z+0.002479}$$

Problem 5.4

Using the properties of the z-transform, find the z-transform for each of the following sequences:

(a) $x(n) = -2u(n) - (0.75)^n u(n)$ (b) $x(n) = e^{-2(n-3)}\sin(0.2\pi(n-3))u(n-3),$ where $u(n-3) = 1$ for $n \geq 3$ while $u(n-3) = 0$ for $n < 3$

solution

(a) For $|z| > \max(1, 0.5) = 1$, z-transform:

$$X(z)=\sum_{n=0}^{\infty}x(n)z^{-n}=\sum_{n=0}^{\infty}-2(z^{-1})^n+(0.75z^{-1})^n=-2\sum_{n=0}^{\infty}(z^{-1})^n+\sum_{n=0}^{\infty}(0.75z^{-1})^n$$
$$=\frac{-2}{1-z^{-1}}+\frac{1}{1-0.75z^{-1}}$$

(b) For $|z|>e^{-2}$, z-transform:

$$X(z)=\sum_{n=0}^{\infty}x(n)z^{-n}=\sum_{n=0}^{\infty}e^{-2(n-3)}\sin(0.2\pi(n-3))u(n-3)z^{-n}$$
$$=z^{-3}\sum_{n=3}^{\infty}e^{-2(n-3)}\sin(0.2\pi(n-3))z^{-(n-3)}$$
$$=z^{-3}\sum_{n=0}^{\infty}e^{-2n}\sin(0.2\pi n)z^{-n}=z^{-3}\frac{1}{j2}[\frac{1}{1-e^{-2}e^{j0.2\pi}z^{-1}}-\frac{1}{1-e^{-2}e^{-j0.2\pi}z^{-1}}]$$
$$=z^{-3}\frac{e^{-2}\sin(0.2\pi)z^{-1}}{1-2e^{-2}\cos(0.2\pi)z^{-1}+e^{-4}z^{-2}}$$
$$=\frac{0.07955z^{-2}}{z^2-0.2190z+0.01832}$$

Problem 5.5

Given two sequences

$$x_1(n)=5\delta(n)-2\delta(n-2)\text{ and }x_2(n)=3\delta(n-3)$$

(a) Determine the z-transform of convolution of the two sequences using the convolution property of z-transform

$$X(z)=X_1(z)X_2(z)$$

(b) Determine the convolution by the inverse z-transform from the result in (a)

$$x(n)=Z^{-1}\left(X_1(z)X_2(z)\right)$$

solution

(a)

$$X_1(z)=\sum_{n=0}^{\infty}[5\delta(n)-2\delta(n-2)]z^{-n}=5-2z^{-2}$$
$$X_2(z)=\sum_{n=0}^{\infty}3\delta(n-3)z^{-n}=3z^{-3}$$
$$X(z)=X_1(z)X_2(z)=(5-2z^{-2})(3z^{-3})=15z^{-3}-6z^{-5}$$

(b)

method 1: Applying the inverse z-transform $Z^{-1}[1]=\delta(n)$ and using the shift theorem $Z^{-1}[z^{-n_0}X(z)]=x(n-n_0)\mathrm{u}(n-n_0)$, having:

$$x(n)=15\delta(n-3)-6\delta(n-5)$$

Problem 5.9

Using the partial fraction expansion method, find the inverse of the following z-transforms:

(a) $X(z)=\frac{1}{z^2-0.3z-0.04}$ (b) $X(z)=\frac{z}{(z-0.2)(z+0.4)}$ (c) $X(z)=\frac{z}{(z+0.2)(z^2-z+0.5)}$ (d) $X(z)=\frac{z(z+0.5)}{(z-0.1)^2(z-0.6)}$

solution

(a)

$$\begin{aligned}\frac{X(z)}{z}&=\frac{-25}{z}+\frac{25z-7.5}{z^2-0.3z-0.04}\\&=\frac{-25}{z}+\frac{\frac{25\times(-0.1)-7.5}{-0.1-0.4}}{z+0.1}+\frac{\frac{25\times0.4-7.5}{0.4+0.1}}{z-0.4}\\&=\frac{-25}{z}+\frac{20}{z+0.1}+\frac{5}{z-0.4}\\X(z)&=-25+20\frac{z}{z+0.1}+5\frac{z}{z-0.4}\\&=-25+20\frac{1}{1+0.1z^{-1}}+5\frac{1}{1-0.4z^{-1}}\\x(n)&=Z^{-1}[X(z)]=-25\delta(n)+[20\cdot(-0.1)^n+5\cdot(0.4)^n]\mathrm{u}(n)\end{aligned}$$

(b)

$$\begin{aligned}\frac{X(z)}{z}&=\frac{1}{(z-0.2)(z+0.4)}=\frac{\frac{1}{0.2+0.4}}{z-0.2}+\frac{\frac{1}{-0.4-0.2}}{z+0.4}=\frac{5}{3}[\frac{1}{z-0.2}-\frac{1}{z+0.4}]\\X(z)&=\frac{5}{3}[\frac{z}{z-0.2}-\frac{z}{z+0.4}]=\frac{5}{3}[\frac{1}{1-0.2z^{-1}}-\frac{1}{1+0.4z^{-1}}]\\x(n)&=Z^{-1}[\frac{5}{3}[\frac{1}{1-0.2z^{-1}}-\frac{1}{1+0.4z^{-1}}]]=\frac{5}{3}[0.2^n-(-0.4)^n]\mathrm{u}(n)=1.667[0.2^n-(-0.4)^n]\mathrm{u}(n)\end{aligned}$$

(c)

$$
\begin{aligned}
\frac{X(z)}{z} &= \frac{1}{(z+0.2)\left(z^2-z+0.5\right)} \\
&= \frac{A}{z+0.2} + \frac{B+jC}{z-(0.5+j0.5)} + \frac{B-jC}{z-(0.5-j0.5)} \\
&= \frac{\frac{1}{(-0.2)^2-(-0.2)+0.5}}{z+0.2} + \frac{\frac{1}{(0.5+j0.5+0.2)(0.5+j0.5-(0.5-j0.5))}}{z-(0.5+j0.5)} + \frac{B-jC}{z-(0.5-j0.5)} \\
&= \frac{\frac{1}{0.74}}{z+0.2} + \frac{\frac{-0.5-j0.7}{0.5^2+0.7^2}}{z-(0.5+j0.5)} + \frac{B-jC}{z-(0.5-j0.5)} \\
&= \frac{\frac{1}{0.74}}{z+0.2} + \frac{\frac{-0.5-j0.7}{0.5^2+0.7^2}}{z-(0.5+j0.5)} + \frac{\frac{-0.5+j0.7}{0.5^2+0.7^2}}{z-(0.5-j0.5)} \\
&= \frac{\frac{1}{0.74}}{z+0.2} + \frac{\frac{-0.5-j0.7}{0.74}}{z-(0.5+j0.5)} + \frac{\frac{-0.5+j0.7}{0.74}}{z-(0.5-j0.5)} \\
X(z) &= \frac{1}{0.74}\frac{1}{1+0.2z^{-1}} + (\frac{-0.5-j0.7}{0.74})\frac{1}{1-(0.5+j0.5)z^{-1}} + (\frac{-0.5+j0.7}{0.74})\frac{1}{1-(0.5-j0.5)z^{-1}} \\
x(n) &= Z^{-1}[X(z)] = [\frac{1}{0.74}(-0.2)^n + (\frac{-0.5-j0.7}{0.74})(0.5+j0.5)^n + (\frac{-0.5+j0.7}{0.74})(0.5-j0.5)^n]\mathrm{u}(n) \\
&= [\frac{1}{0.74}(-0.2)^n + 2\mathrm{Re}\left\{(\frac{-0.5-j0.7}{0.74})(0.5+j0.5)^n\right\}]\mathrm{u}(n) \\
&= [\frac{1}{0.74}(-0.2)^n + 2\mathrm{Re}\left\{e^{j(-2.191)}(\frac{1}{\sqrt{2}}e^{j0.25\pi})^n\right\}]\mathrm{u}(n) \\
&= [\frac{1}{0.74}(-0.2)^n + 2\mathrm{Re}\left\{(\frac{1}{\sqrt{2}})^n e^{j(0.25\pi n-2.191)}\right\}]\mathrm{u}(n) \\
&= [\frac{1}{0.74}(-0.2)^n + 2(\frac{1}{\sqrt{2}})^n\cos(0.25\pi n-2.191)]\mathrm{u}(n) \\
&= [1.351(-0.2)^n + 2(0.7071)^n\cos(0.25\pi n-2.191)]\mathrm{u}(n) \\
&= [1.351(-0.2)^n + 2(0.7071)^n\cos(45^\circ n-125.5^\circ)]\mathrm{u}(n)
\end{aligned}
$$

(d)

$$
\begin{aligned}
\frac{X(z)}{z} &= \frac{z+0.5}{(z-0.1)^2(z-0.6)} \\
&= \frac{d(\frac{z+0.5}{z-0.6})/dz|_{z=0.1}}{z-0.1} + \frac{\frac{0.1+0.5}{0.1-0.6}}{(z-0.1)^2} + \frac{\frac{0.6+0.5}{(0.6-0.1)^2}}{z-0.6} \\
&= \frac{\frac{-1.1}{(0.1-0.6)^2}}{z-0.1} + \frac{(-1.2)}{(z-0.1)^2} + \frac{4.4}{z-0.6} \\
&= \frac{(-4.4)}{z-0.1} + \frac{(-1.2)}{(z-0.1)^2} + \frac{4.4}{z-0.6} \\
X(z) &= (-4.4)\frac{z}{z-0.1} + (-1.2)\frac{z}{(z-0.1)^2} + 4.4\frac{z}{z-0.6} \\
&= (-4.4)\frac{1}{1-0.1z^{-1}} + (-1.2/0.1)\frac{0.1z^{-1}}{(1-0.1z^{-1})^2} + 4.4\frac{1}{1-0.6z^{-1}} \\
x(n) &= Z^{-1}[X(z)] \\
&= [(-4.4)(0.1)^n + (-12)n(0.1)^n + 4.4\cdot(0.6)^n]\mathrm{u}(n)
\end{aligned}
$$

Problem 5.12

A system is described by the difference equation

$$y(n) + 0.2y(n-1) = 4(0.3)^n u(n)$$

Determine the solution when the initial condition is $y(-1) = 1$.

solution

$$\begin{aligned}
\sum_{n=0}^{\infty}[y(n) + 0.2y(n-1)]z^{-n} &= \sum_{n=0}^{\infty} y(n)z^{-n} + 0.2\sum_{n=0}^{\infty} y(n-1)z^{-n} \\
&= \sum_{n=0}^{\infty} y(n)z^{-n} + 0.2z^{-1}\sum_{n=-1}^{\infty} y(n)z^{-n} \\
&= Y(z) + 0.2z^{-1}[y(-1)z + Y(z)] = [1 + 0.2z^{-1}]Y(z) + 0.2y(-1) \\
&= [1 + 0.2z^{-1}]Y(z) + 0.2 \quad [y(-1) = 1] \\
&= \sum_{n=0}^{\infty} 4(0.3)^n u(n) = \frac{4}{1 - 0.3z^{-1}} \\
Y(z) &= \frac{\frac{4}{1-0.3z^{-1}} - 0.2}{1 + 0.2z^{-1}} = \frac{\frac{3.8+0.06z^{-1}}{1-0.3z^{-1}}}{1 + 0.2z^{-1}} \\
&= \frac{3.8 + 0.06z^{-1}}{(1 - 0.3z^{-1})(1 + 0.2z^{-1})} \\
\frac{Y(z)}{z} &= \frac{3.8z + 0.06}{(z - 0.3)(z + 0.2)} = \frac{\frac{3.8\times0.3+0.06}{0.3+0.2}}{z - 0.3} + \frac{\frac{3.8\times(-0.2)+0.06}{-0.2-0.3}}{z + 0.2} \\
&= \frac{2.4}{z - 0.3} + \frac{1.4}{z + 0.2} \\
Y(z) &= 2.4\frac{1}{1 - 0.3z^{-1}} + 1.4\frac{1}{1 + 0.2z^{-1}} \\
y(n) &= Z^{-1}[Y(z)] = [2.4 \cdot (0.3)^n + 1.4 \cdot (-0.2)^n]\mathrm{u}(n)
\end{aligned}$$

Finally

$$y(n) = [2.4 \cdot (0.3)^n + 1.4 \cdot (-0.2)^n]\mathrm{u}(n)$$

Problem 5.14

Given the following difference equation with the input-output relationship of a certain initially relaxed system (all initial conditions are zero),

$$y(n) - 0.7y(n-1) + 0.1y(n-2) = x(n) + x(n-1)$$

(a) Find the impulse response sequence y(n) due to the impulse sequence δ(n). (b) Find the output response of the system when the unit step function u(n) is applied.

solution

(a) Because all initial conditions are zero:

$$Y(z) - 0.7z^{-1}Y(z) + 0.1z^{-1}Y(z) = Z[\delta(n) + \delta(n-1)] = 1 + z^{-1}$$

$$Y(z) = \frac{1 + z^{-1}}{1 - 0.7z^{-1} + 0.1z^{-2}}$$

$$\begin{aligned} \frac{Y(z)}{z} &= \frac{z+1}{z^2 - 0.7z + 0.1} = \frac{\frac{0.2+1}{0.2-0.5}}{z-0.2} + \frac{\frac{0.5+1}{0.5-0.2}}{z-0.5} \\ &= \frac{(-4)}{z-0.2} + \frac{5}{z-0.5} \\ Y(z) &= (-4)\frac{z}{z-0.2} + 5\frac{z}{z-0.5} = (-4)\frac{1}{1-0.2z^{-1}} + 5\frac{1}{1-0.5z^{-1}} \\ y(n) &= Z^{-1}[Y(z)] = [-4 \cdot (0.2)^n + 5 \cdot (0.5)^n]\mathrm{u}(n) \end{aligned}$$

(b) Here we know the impulse response of system is $h(n) = [-4 \cdot (0.2)^n + 5 \cdot (0.5)^n]\mathrm{u}(n)$, then:

$$\begin{aligned} y(n) = h(n) * \mathrm{u}(n) &= \sum_{n'=-\infty}^{\infty} h(n')\mathrm{u}(n-n') \\ &= \sum_{n'=-\infty}^{\infty} [-4 \cdot (0.2)^{n'} + 5 \cdot (0.5)^{n'}]\mathrm{u}(n')\mathrm{u}(n-n') \\ &= \sum_{n'=0}^{n} [-4 \cdot (0.2)^{n'} + 5 \cdot (0.5)^{n'}] \\ &= \left[(-4)\frac{1 - 0.2 \cdot (0.2)^n}{1-0.2} + 5\frac{1 - 0.5 \cdot (0.5)^n}{1-0.5}\right]\mathrm{u}(n) \\ &= \left[(-5)[1 - 0.2 \cdot (0.2)^n] + 10[1 - 0.5 \cdot (0.5)^n]\right]\mathrm{u}(n) \\ &= [5 + (0.2)^n - 5 \cdot (0.5)^n]\mathrm{u}(n) \end{aligned}$$

Problem 5.19

Use the initial and final value theorems to find x(0) and x(∞) for Problem 5.11(a), (b), (d).

(a) $X(z) = \frac{1}{z^2+0.5z+0.06}$ (b) $X(z) = \frac{z}{(z+0.3)(z-0.5)}$ (d) $X(z) = \frac{2z(z-0.4)}{(z-0.2)^2(z+0.8)}$

solution

Initial value theorem:

$$\begin{aligned} \lim_{z\to+\infty} X(z) = \lim_{z\to+\infty} \sum_{n=0}^{\infty} x(n)z^{-n} &= \sum_{n=0}^{\infty} x(n) \lim_{z\to+\infty} z^{-n} \\ &= \sum_{n=0}^{\infty} x(n)\delta(n) = x(0) \end{aligned}$$

Final value theorem(z=1 is in ROC of $X(z)$)

$$
\begin{aligned}
\lim_{z\to 1}(z-1)X(z) &= \lim_{z\to 1}\lim_{N\to\infty}[z\sum_{n=0}^{N+1}x(n)z^{-n} - \sum_{n=0}^{N}x(n)z^{-n}] \\
&= \lim_{z\to 1}\lim_{N\to\infty}[\sum_{n=0}^{N}x(n)(z-1)z^{-n} + x(N+1)z^{-N}] \\
&= \lim_{N\to\infty}\lim_{z\to 1}[\sum_{n=0}^{N}x(n)(z-1)z^{-n} + x(N+1)z^{-N}] \\
&= \lim_{N\to\infty} x(N+1) \\
&= x(+\infty)
\end{aligned}
$$

(a)

$$
\begin{aligned}
x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{1}{z^2+0.5z+0.06} = 0 \\
x(+\infty) = \lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1}(z-1)\frac{1}{z^2+0.5z+0.06} = 0
\end{aligned}
$$

(b)

$$
\begin{aligned}
x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{z}{(z+0.3)(z-0.5)} = 0 \\
x(+\infty) = \lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1}(z-1)\frac{z}{(z+0.3)(z-0.5)} = 0
\end{aligned}
$$

(d)

$$
\begin{aligned}
x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{2z(z-0.4)}{(z-0.2)^2(z+0.8)} = 0 \\
x(+\infty) = \lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1}(z-1)\frac{2z(z-0.4)}{(z-0.2)^2(z+0.8)} = 0
\end{aligned}
$$

Problem 5.20

Use power series method to find x(0), x(1), x(2), x(3), x(4) for Problem 5.11(a), (b).

(b) $X(z) = \frac{z}{(z+0.3)(z-0.5)}$

solution

$$
X(z) = \frac{z}{(z+0.3)(z-0.5)} = \frac{z}{z^2-0.2z-0.15}
$$

Using the long division method, we yield

$$
\begin{array}{r|llllll}
 & z^{-1} & +0.2z^{-2} & +0.19z^{-3} & +0.068z^{-4} & \cdots & \\
\hline
z^2-0.2z-0.15 & z & & & & & \\
 & z & -0.2 & -0.15z^{-1} & & & \\
\hline
 & & 0.2 & +0.15z^{-1} & & & \\
 & & 0.2 & -0.04z^{-1} & -0.03z^{-2} & & \\
\hline
 & & & 0.19z^{-1} & +0.03z^{-2} & & \\
 & & & 0.19z^{-1} & -0.038z^{-2} & -0.0285z^{-3} & \\
\hline
 & & & & 0.068z^{-2} & +0.0285z^{-3} & \\
 & & & & 0.068z^{-2} & -0.01368z^{-3} & -0.0102z^{-4} \\
\hline
 & & & & & 0.04218z^{-3} & +0.0102z^{-4}
\end{array}
$$

This leads to

$$X(z) = 0 + 1 \cdot z^{-1} + 0.2 \cdot z^{-2} + 0.19 \cdot z^{-3} + 0.068 \cdot z^{-4} + \cdots$$

We see that:

$$
\begin{aligned}
x(0) &= 0 \\
x(1) &= 1 \\
x(2) &= 0.2 \\
x(3) &= 0.19 \\
x(4) &= 0.068
\end{aligned}
$$

Problem 5.21

Use the residue formula to find the inverse of the z-transform for Problem 5.11(a), (b), (d).

(b) $X(z) = \frac{z}{(z+0.3)(z-0.5)}$

solution

(b) C is any simple closed curve in Region of Convergence

$$x(n) = \frac{1}{j2\pi} \oint_C X(z) z^{n-1} dz \quad [\text{C is in ROC: } |z| > 0.5]$$

Let C to be a circle $z = Re^{j\theta}, [R > 0.5, \theta \in [0, 2\pi)]$, then:

$$
\begin{aligned}
x(n) &= \frac{1}{j2\pi} \oint_{z=Re^{j\theta}} X(z) z^{n-1} dz \\
&= \text{Res}[X(z)z^{n-1}, -0.3] + \text{Res}[X(z)z^{n-1}, 0.5] + \text{Res}[X(z)z^{n-1}, 0] \\
&= -\text{Res}[X(z)z^{n-1}, \infty]
\end{aligned}
$$

Notice,

$$\sum_{p_i} \mathrm{Res}[f(z), p_i] + \mathrm{Res}[f(z), \infty] = 0$$

$$\mathrm{Res}[f(z), \infty] \equiv -\frac{1}{j2\pi} \oint_{Re^{i\theta}} f(z)dz \quad [\forall p_i, |p_i| < R]$$

$$= -\mathrm{Res}[f(\frac{1}{\xi})\frac{1}{\xi^2}, 0] \quad [\xi \equiv \frac{1}{z}]$$

For $n \geq 0$, having:

$$\begin{aligned} x(n) &= \mathrm{Res}[X(z)z^{n-1}, -0.3] + \mathrm{Res}[X(z)z^{n-1}, 0.5] + \mathrm{Res}[X(z)z^{n-1}, 0] \\ &= \mathrm{Res}[\frac{z^n}{(z+0.3)(z-0.5)}, -0.3] + \mathrm{Res}[\frac{z^n}{(z+0.3)(z-0.5)}, 0.5] + 0 \\ &= \frac{z \cdot z^{n-1}}{z-0.5}|_{z=-0.3} + \frac{z \cdot z^{n-1}}{z+0.3}|_{z=0.5} \\ &= \frac{1}{-0.8}(-0.3)^n + \frac{1}{0.8}(0.5)^n \\ &= -1.25 \cdot (-0.3)^n + 1.25 \cdot (0.5)^n \end{aligned}$$

For $n < 0$, having:

$$\begin{aligned} x(n) &= -\mathrm{Res}[X(z)z^{n-1}, \infty] \\ &= -\mathrm{Res}[\frac{z^n}{(z+0.3)(z-0.5)}, \infty] \\ &= \mathrm{Res}[f(\frac{1}{\xi})\frac{1}{\xi^2}, 0] \quad [\xi \equiv \frac{1}{z}, f(z) = \frac{z^n}{(z+0.3)(z-0.5)}] \\ &= \mathrm{Res}[\frac{\xi^{-n}}{(\frac{1}{\xi}+0.3)(\frac{1}{\xi}-0.5)}\frac{1}{\xi^2}, 0] \\ &= \mathrm{Res}[\frac{\xi^{-n}}{(1+0.3\xi)(1-0.5\xi)}, 0] \\ &= 0 \end{aligned}$$

To sum up:

$$x(n) = [-1.25 \cdot (-0.3)^n + 1.25 \cdot (0.5)^n]\mathrm{u}(n)$$

Advanced Problems

Problem 5.22

If $y(n) = e^{-an}x(n)$, where $a \geq 0$ and $n \geq 0$, show that

$$Y(z) = X(ze^a)$$

solution

When z is in ROC, $|z'| = |z| \cdot |e^a| \geq |z|$ in ROC, too

$$
\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} x(n) z^{-n} \\
Y(z) &= \sum_{n=0}^{\infty} e^{-an} x(n) z^{-n} \\
&= \sum_{n=0}^{\infty} x(n) (ze^{a})^{-n} \\
&= \sum_{n=0}^{\infty} x(n) (z')^{-n} \quad [z' = ze^{a}] \\
&= X(z') = X(ze^{a})
\end{aligned}
$$

So, we conclude that:

$$Y(z) = X\left(ze^{a}\right)$$

Problem 5.23

If $y(n) = nx(n)$, where $a \geq 0$ and $n \geq 0$, show that

$$Y(z) = -z\frac{dX(z)}{dz}$$

solution

$$
\begin{aligned}
-z\frac{dX(z)}{dz} &= (-z)\frac{d}{dz}[\sum_{n=0}^{\infty} x(n) z^{-n}] \\
&= (-z)\sum_{n=0}^{\infty} x(n)\frac{d}{dz}[z^{-n}] \\
&= (-z)\sum_{n=0}^{\infty} x(n)(-n) z^{-n-1} \quad [n \geq 0] \\
&= \sum_{n=0}^{\infty} n \cdot x(n) z^{-n} \\
&= \sum_{n=0}^{\infty} y(n) z^{-n} = Z[y(n)] \\
&= Y(z)
\end{aligned}
$$

So, we verify that:

$$Y(z) = -z\frac{dX(z)}{dz}$$

Problem 5.30

Given

$$X(z) = \frac{(a-b)}{\left(1-az^{-1}\right)(z-b)} \text{ and } a < |z| < b, 0 < a < 1 \text{ and } b > 1$$

use the inversion formula to show that

$$x(n) = \begin{cases} b^n & n < 0 \\ a^n & n \geq 0 \end{cases}$$

solution

Set C to a simple curve in ROC $a < |z| < b$, then:

$$\begin{aligned} x(n) &= \frac{1}{j2\pi} \oint_C X(z) z^{n-1} dz \\ &= \text{Res}[X(z)z^{n-1}, a] + \text{Res}[X(z)z^{n-1}, 0] \\ &= -\left\{ \text{Res}[X(z)z^{n-1}, b] + \text{Res}[X(z)z^{n-1}, \infty] \right\} \end{aligned}$$

Note,

$$\sum_{p_i} \text{Res}[f(z), p_i] + \text{Res}[f(z), \infty] = 0$$

$$\begin{aligned} \text{Res}[f(z), \infty] &\equiv -\frac{1}{j2\pi} \oint_{Re^{i\theta}} f(z) dz \quad [\forall p_i, |p_i| < R] \\ &= -\text{Res}[f(\frac{1}{\xi})\frac{1}{\xi^2}, 0] \quad [\xi \equiv \frac{1}{z}] \end{aligned}$$

Here:

$$\begin{aligned} \text{Res}[X(z)z^{n-1}, a] &= \frac{z-a}{0!} \frac{(a-b)z^{n-1}}{(1-az^{-1})(z-b)}|_{z=a} \\ &= \frac{(a-b)z^n}{(z-b)}|_{z=a} \\ &= a^n \\ \text{Res}[X(z)z^{n-1}, b] &= \frac{z-b}{0!} \frac{(a-b)z^{n-1}}{(1-az^{-1})(z-b)}|_{z=b} \\ &= \frac{(a-b)z^n}{(z-a)}|_{z=b} \\ &= -b^n \end{aligned}$$

Moreover:

$$
\begin{aligned}
\mathrm{Res}[X(z)z^{n-1},0] &= \mathrm{Res}[\frac{(a-b)z^n}{(z-a)(z-b)},0] \\
&= \begin{cases} \frac{d^{-n-1}}{dz^{-n-1}}[\frac{z^{-n}}{(-n-1)!}\frac{(a-b)z^n}{(z-a)(z-b)}]|_{z=0} & n<0 \\ 0 & n\geq 0 \end{cases} \\
&= \begin{cases} \frac{d^{-n-1}}{dz^{-n-1}}[\frac{1}{(-n-1)!}\frac{(a-b)}{(z-a)(z-b)}]|_{z=0} & n<0 \\ 0 & n\geq 0 \end{cases} \\
&= \begin{cases} \frac{1}{(-n-1)!}\frac{d^{-n-1}}{dz^{-n-1}}[\frac{1}{(z-a)}-\frac{1}{(z-b)}]|_{z=0} & n<0 \\ 0 & n\geq 0 \end{cases} \\
&= \begin{cases} \frac{1}{(-n-1)!}[\frac{(-1)^{-n-1}(-n-1)!}{(z-a)^{-n}}-\frac{(-1)^{-n-1}(-n-1)!}{(z-b)^{-n}}]|_{z=0} & n<0 \\ 0 & n\geq 0 \end{cases} \\
&= \begin{cases} (-1)[\frac{1}{(-z+a)^{-n}}-\frac{1}{(-z+b)^{-n}}]|_{z=0} & n<0 \\ 0 & n\geq 0 \end{cases} \\
&= \begin{cases} (-1)[a^n-b^n] & n<0 \\ 0 & n\geq 0 \end{cases} \\
\mathrm{Res}[X(z)z^{n-1},\infty] &= -\mathrm{Res}[X(\frac{1}{\xi})(\frac{1}{\xi})^{n-1}\frac{1}{\xi^2},0] \quad [\xi\equiv\frac{1}{z}] \\
&= -\mathrm{Res}[\frac{(a-b)}{(1-a\xi)\,(\frac{1}{\xi}-b)}(\frac{1}{\xi})^{n+1},0] \\
&= -\mathrm{Res}[\frac{(a-b)}{(1-a\xi)\,(1-b\xi)\xi^n},0] \\
&= \begin{cases} 0 & n\leq 0 \\ -\frac{d^{n-1}}{d\xi^{n-1}}[\frac{\xi^n}{(n-1)!}\frac{(a-b)}{(1-a\xi)(1-b\xi)\xi^n}]|_{\xi=0} & n>0 \end{cases} \\
&= \begin{cases} 0 & n\leq 0 \\ -\frac{1}{(n-1)!}\frac{d^{n-1}}{d\xi^{n-1}}[\frac{(a-b)}{(1-a\xi)(1-b\xi)}]|_{\xi=0} & n>0 \end{cases} \\
&= \begin{cases} 0 & n\leq 0 \\ -\frac{1}{(n-1)!}\frac{d^{n-1}}{d\xi^{n-1}}[\frac{1}{(1/a-\xi)}-\frac{1}{(1/b-\xi)}]|_{\xi=0} & n>0 \end{cases} \\
&= \begin{cases} 0 & n\leq 0 \\ -\frac{1}{(n-1)!}(n-1)![\frac{1}{(1/a-\xi)^n}-\frac{1}{(1/b-\xi)^n}]|_{\xi=0} & n>0 \end{cases} \\
&= \begin{cases} 0 & n\leq 0 \\ [-\frac{1}{(1/a-\xi)^n}+\frac{1}{(1/b-\xi)^n}]|_{\xi=0} & n>0 \end{cases} \\
&= \begin{cases} 0 & n\leq 0 \\ (-1)[a^n-b^n] & n>0 \end{cases}
\end{aligned}
$$

So, for $n\geq 0$, having:

$$
\begin{aligned}
x(n) &= \frac{1}{j2\pi}\oint_C X(z)z^{n-1}dz \\
&= \mathrm{Res}[X(z)z^{n-1},a]+\mathrm{Res}[X(z)z^{n-1},0] \\
&= a^n+0 \\
&= -\left\{\mathrm{Res}[X(z)z^{n-1},b]+\mathrm{Res}[X(z)z^{n-1},\infty]\right\} \\
&= -\{-b^n+(-1)[a^n-b^n]\} \\
&= a^n
\end{aligned}
$$

For $n<0$, having:

$$
\begin{aligned}
x(n) &= \frac{1}{j2\pi}\oint_C X(z)z^{n-1}dz \\
&= \mathrm{Res}[X(z)z^{n-1}, a] + \mathrm{Res}[X(z)z^{n-1}, 0] \\
&= a^n + (-1)[a^n - b^n] \\
&= -\left\{\mathrm{Res}[X(z)z^{n-1}, b] + \mathrm{Res}[X(z)z^{n-1}, \infty]\right\} \\
&= -\{-b^n + 0\} \\
&= b^n
\end{aligned}
$$

To sum up, we have:

$$
x(n) = \begin{cases} b^n & n < 0 \\ a^n & n \geq 0 \end{cases}
$$