Homework 2

Course Title: Digital Signal Processing I (Spring 2020)

Course Number: ECE53800

Instructor: Dr. Li Tan

Author: **Zhankun Luo**

Problem: Problems 3.2: a, c, 3.6, 3.9: a,b, 3.10, 3.13, 3.15, 3.18: a, b, 3.23, 3.27, 3.28

3.30: b, d, 3.31 c, 3.34

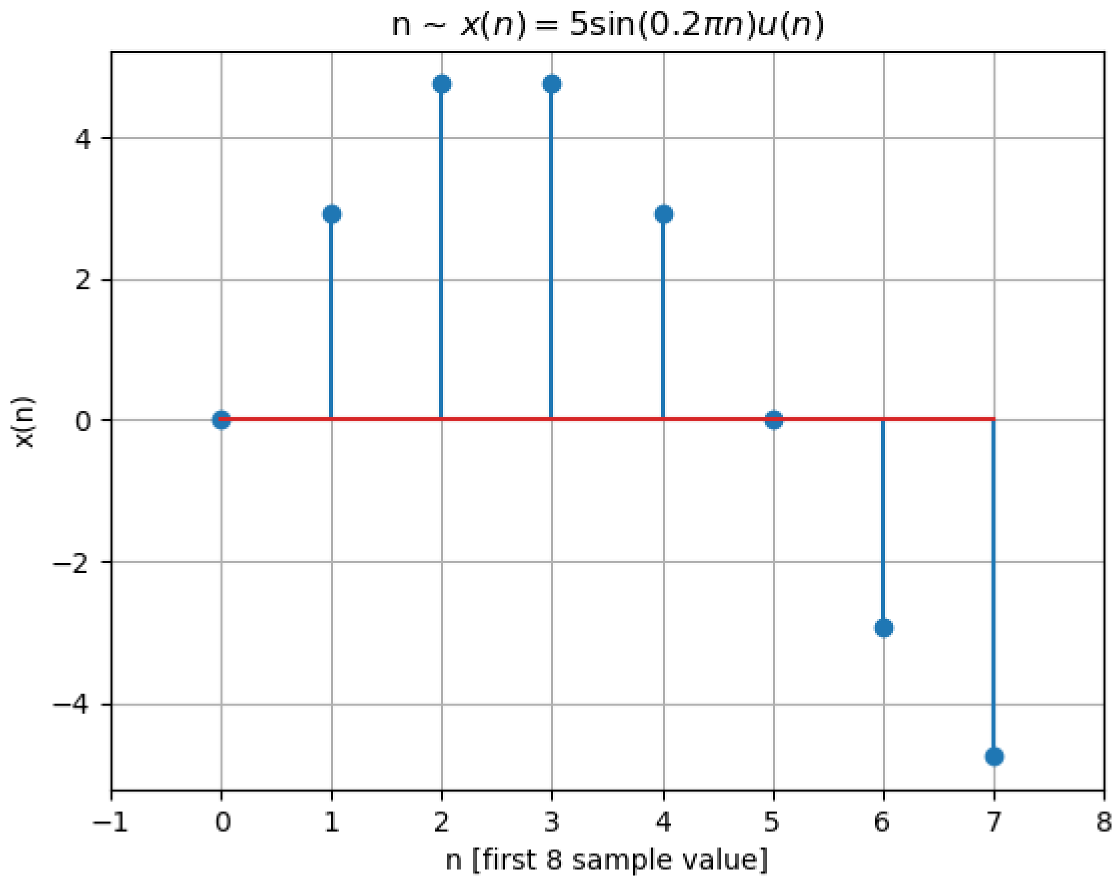
Problems

Problem 3.2

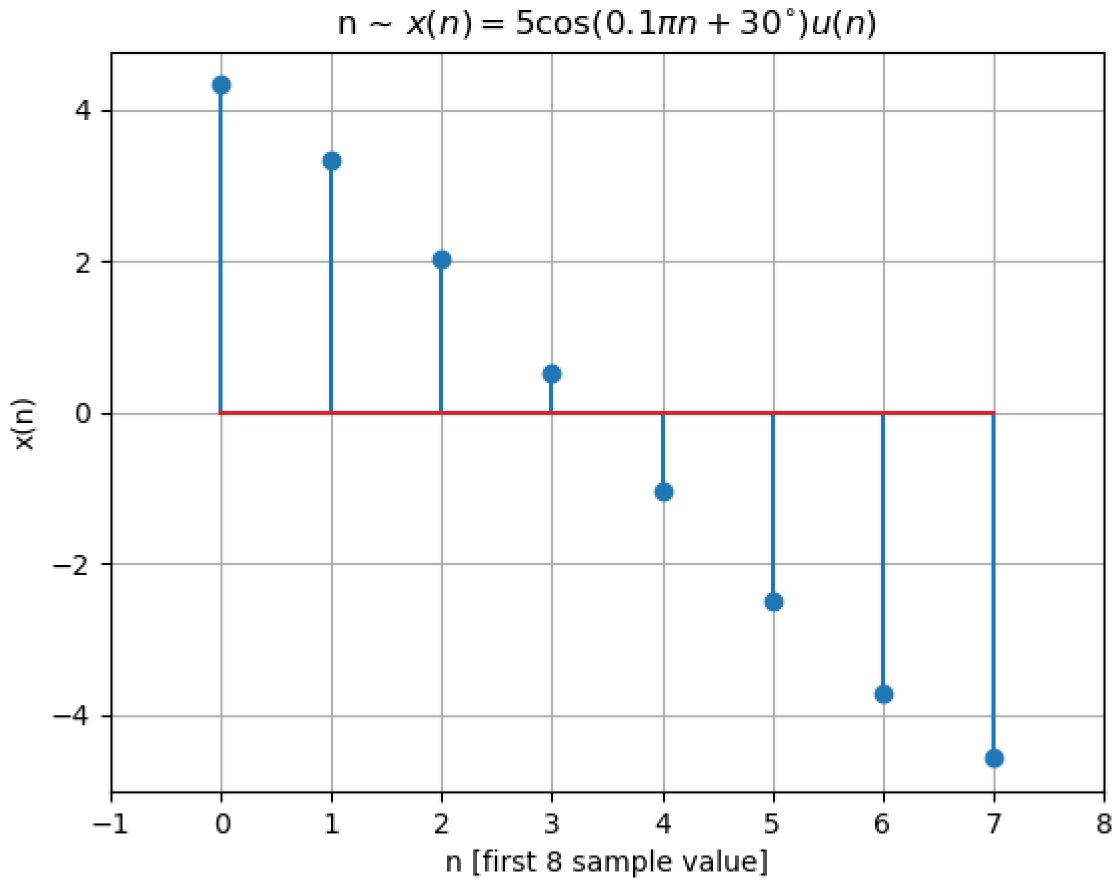
Calculate the first eight sample values and sketch each of the following sequences:

$$\begin{aligned}(a) \quad & x(n) = 5\sin(0.2\pi n)u(n) \\ (c) \quad & x(n) = 5\cos(0.1\pi n + 30^\circ)u(n)\end{aligned}$$

solution



n ~ $x(n) = 5\sin(0.2\pi n)u(n)$
x(n)
4
2
0
−2
−4
−1
0
1
2
3
4
5
6
7
8
n [first 8 sample value]



n ~ $x(n) = 5\cos(0.1\pi n + 30^{\circ})u(n)$
x(n)
4
2
0
−2
−4
−1
0
1
2
3
4
5
6
7
8
n [first 8 sample value]

Problem 3.6

Given the digital signals x(n) in Figs. 3.24 and 3.25, write an expression for each digital signal using the unit-impulse sequence and its shifted sequences.

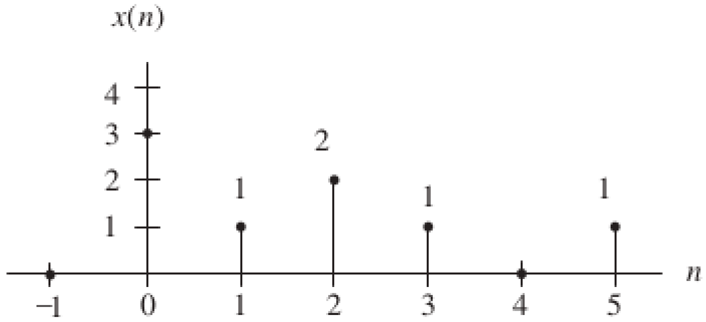


FIG. 3.24

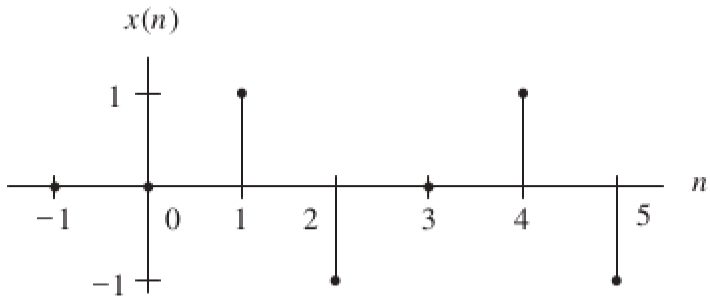


FIG. 3.25

solution

1. Fig 3.24

$$x(n) = 3\delta(n) + \delta(n-1) + 2\delta(n-2) + \delta(n-3) + \delta(n-5)$$

2. Fig 3.25

$$x(n) = \delta(n-1) - \delta(n-2) + \delta(n-4) - \delta(n-5)$$

Problem 3.9

Assume that a digital signal processor with a sampling time interval of 0.01 s converts each of the following analog signals x(t) to a digital signal x(n), determine the digital sequences for each of the following analog signals.

$$\begin{aligned} &(a) \quad x(t) = e^{-50t}u(t) \\ &(b) \quad x(t) = 5\sin(20\pi t)u(t) \end{aligned}$$

solution

$T = 0.01$, then $x(n) \equiv x(t)|_{t=nT}$, so having:

$$\begin{aligned} &(a) \quad x(n) = e^{-50t}u(t)|_{t=nT} = e^{-50Tn}u(nT) = e^{-0.5n}u(n) \\ &(b) \quad x(n) = 5\sin(20\pi t)u(t)|_{t=nT} = 5\sin(20\pi Tn)u(nT) = 5\sin(0.2\pi n)u(n) \end{aligned}$$

Problem 3.10

Determine which of the following systems is a linear system.

$$\begin{aligned} (a) &\quad y(n) = 5x(n) + 2x^2(n) \\ (b) &\quad y(n) = x(n-1) + 4x(n) \\ (c) &\quad y(n) = 4x^3(n-1) - 2x(n) \end{aligned}$$

solution

1. (a) Nonlinear system:

   Example: set $x_1(n) = -1, x_2(n) = 1, x(n) \equiv x_1(n) + x_2(n) = 0$

   Having $y_1(n) = -5 + 2 = -3, y_2(n) = 5 + 2 = 7, y(n) = 0 + 0 = 0$

   Here $y(n) \neq y_1(n) + y_2(n)$
2. (b) Linear system:

   For any input $x_1(n), x_2(n)$, and $a, b \in R$

   set $x(n) \equiv ax_1(n) + bx_2(n)$

   we have

$$\begin{aligned} y_1(n) &= x_1(n-1) + 4x_1(n) \\ y_2(n) &= x_2(n-1) + 4x_2(n) \\ y(n) = x(n-1) + 4x(n) &= [ax_1(n-1) + bx_2(n-1)] + 4[ax_1(n) + bx_2(n)] \\ &= a[x_1(n-1) + 4x_1(n)] + b[x_2(n-1) + 4x_2(n)] = ay_1(n) + by_2(n) \end{aligned}$$

3. (c) Nonlinear system:

   Example: set $x_1(n) = 2, x_2(n) = 1, x(n) \equiv x_1(n) + x_2(n) = 3$

   Having $y_1(n) = 28, y_2(n) = 2, y(n) = 102$

   Here $y(n) \neq y_1(n) + y_2(n)$

Problem 3.13

Given the following linear systems, find which one is time invariant.

$$\begin{aligned} (a) &\quad y(n) = -5x(n-10) \\ (b) &\quad y(n) = 4x(n^2) \end{aligned}$$

solution

1. (a) time invariant

   For input $x_1(n)$, the shifted input of n samples $x_2(n) \equiv x_1(n - n_0)$

$$\begin{aligned} y_1(n) &= -5x_1(n-10) \\ y_1(n-n_0) &= -5x_1(n-n_0-10) \quad [\text{replacing n by } n-n_0] \\ y_2(n) = -5x_2(n-10) &= -5x_1(n-n_0-10) \quad [\text{replacing } x_2(n-10) \text{ by } x_1(n-n_0-10)] \end{aligned}$$

   So, having

$$y_1(n-n_0) = y_2(n)$$

2. (b) time varying

   For input $x_1(n)$, the shifted input of n samples $x_2(n) \equiv x_1(n - n_0)$

$$y_1(n) = 4x_1(n^2)$$
$$y_1(n - n_0) = 4x_1((n - n_0)^2) \quad [\text{replacing n by } n - n_0]$$
$$y_2(n) = 4x_2(n^2) = 4x_1(n^2 - n_0) \quad [\text{replacing } x_2(n^2) \text{ by } x_1(n^2 - n_0)]$$

   So, having

$$y_1(n - n_0) \neq y_2(n)$$

Problem 3.15

Determine the causality for each of the following linear systems.

$$\begin{array}{ll} (a) & y(n) = 0.5x(n) + 20x(n-2) - 0.1y(n-1) \\ (b) & y(n) = x(n+2) - 0.4y(n-1) \\ (c) & y(n) = x(n-1) + 0.5y(n+2) \end{array}$$

solution

(a) Causal; (b) Noncausal; (c) Causal

(a) Since the output y(n) depends on the current input x(n) and the past input x(n-2) , and past output y(n-1), the system is causal.

(b) Since the output y(n) depends on the future input x(n+2) , the system is noncausal.

(c) We can rewrite the formula:

$$0.5y(n) = y(n-2) - x(n-3)$$
$$y(n) = 2y(n-2) - 2x(n-3)$$

Since the output y(n) depends on the past input x(n-3) and the past output y(n-2), the system is causal.

Problem 3.18

Find the unit-impulse response for each of the following linear systems.

$$\begin{array}{ll} (a) & y(n) = 0.2x(n) - 0.3x(n-2); \text{ for } n \geq 0, x(-2) = 0, x(-1) = 0 \\ (b) & y(n) = 0.5y(n-1) + 0.5x(n); \text{ for } n \geq 0, y(-1) = 0 \end{array}$$

solution

1. (a) the unit-impulse response, by replacing $x(n), x(n-2), y(n)$ with $\delta(n), \delta(n-2), h(n)$, having:

$$h(n) = 0.2\delta(n) - 0.3\delta(n-2) \quad [n \geq 0]$$

2. (b) the unit-impulse response, by replacing $x(n), y(n-1), y(n)$ with $\delta(n), h(n-1), h(n)$, having:

$$h(n) = 0.5h(n-1) + 0.5\delta(n) \quad [n \geq 0, h(-1) = 0]$$

Mathematical induction

1. **The initial case:** We know that $h(-1) = 0$

   $h(0) = 0.5 \cdot 0 + 0.5\delta(0) = 0.5$

   So having $h(0) = 0.5 = 0.5^{0+1}$

2. **The induction step:** Then from $h(n) = 0.5h(n-1)\ [n-1 \geq 0]$

   When $h(n-1) = 0.5^n\ [n-1 \geq 0]$, we can infer that

$$h(n-1) = 0.5^n\ [n-1 \geq 0] \Longrightarrow h(n) = ah(n-1) = 0.5^{n+1}\ [n \geq 0]$$

Conclusion: $h(n) = 0.5^{n+1}\ [n \geq 0]$

Problem 3.23

$$h(k) = \begin{cases} 2, & k = 0,1,2 \\ 1, & k = 3,4 \\ 0 & \text{elsewhere} \end{cases} \quad \text{and } x(k) = \begin{cases} 2, & k = 0 \\ 1, & k = 1,2 \\ 0 & \text{elsewhere} \end{cases}$$

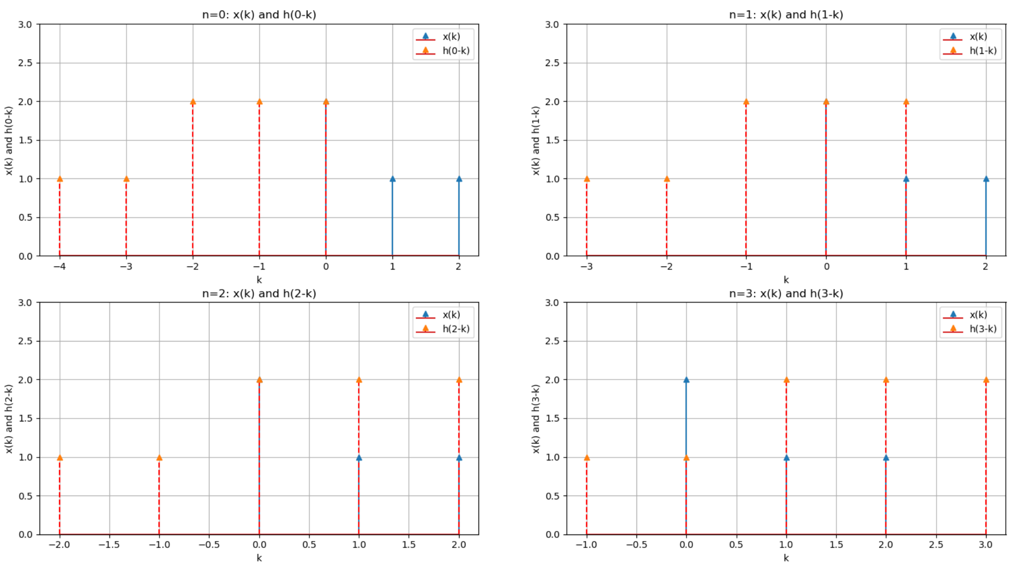
evaluate the digital convolution

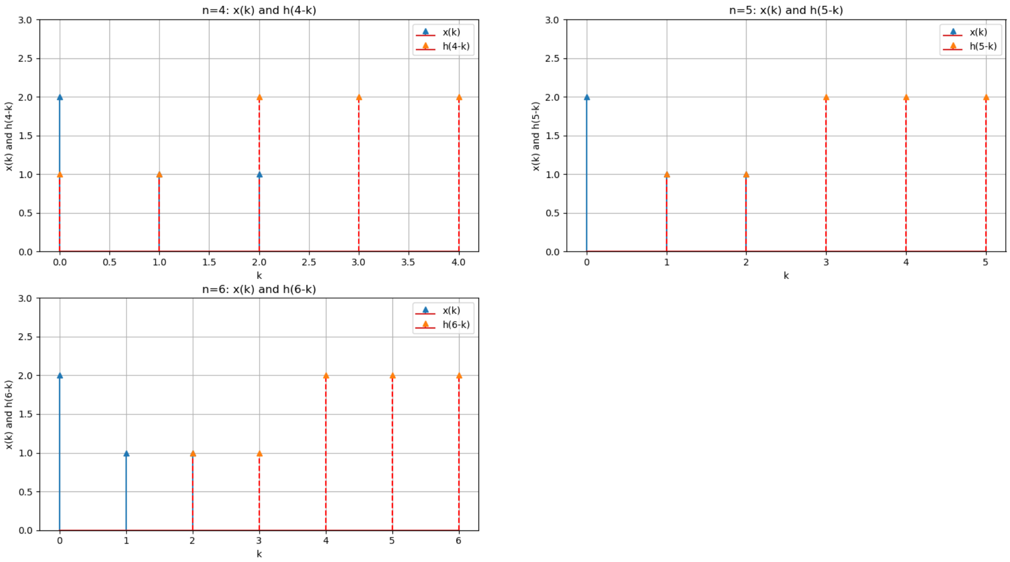
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

1. the graphical method;
2. the table method;
3. applying the convolution formula directly

solution

1. the graphical method;





So we have:

$$y(n) = 4\delta(n) + 6\delta(n-1) + 8\delta(n-2) + 6\delta(n-3) + 5\delta(n-4) + 2\delta(n-5) + \delta(n-6)$$

2. the table method;

k	-4	-3	-2	-1	0	1	2	3	4	5	6
$x(k)$					2	1	1				
$h(0-k)$	1	1	2	2	2						
$h(1-k)$		1	1	2	2	2					
$h(2-k)$			1	1	2	2	2				
$h(3-k)$				1	1	2	2	2			
$h(4-k)$					1	1	2	2	2		
$h(5-k)$						1	1	2	2	2	
$h(6-k)$							1	1	2	2	2

So we have

$$y(n) = 4\delta(n) + 6\delta(n-1) + 8\delta(n-2) + 6\delta(n-3) + 5\delta(n-4) + 2\delta(n-5) + \delta(n-6)$$

3. applying the convolution formula directly

$$h(k) = 2\delta(k) + 2\delta(k-1) + 2\delta(k-2) + \delta(k-3) + \delta(k-4)$$
$$x(k) = 2\delta(k) + \delta(k-1) + \delta(k-2)$$
$$x(n-k) = 2\delta(n-k) + \delta(n-k-1) + \delta(n-k-2)$$

So we have

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
&= \sum_{k=-\infty}^{\infty} [2\delta(k) + 2\delta(k-1) + 2\delta(k-2) + \delta(k-3) + \delta(k-4)]x(n-k) \\
&= 2x(n) + 2x(n-1) + 2x(n-2) + x(n-3) + x(n-4) \\
&= 2[2\delta(n) + \delta(n-1) + \delta(n-2)] + 2[2\delta(n-1) + \delta(n-2) + \delta(n-3)] \\
&\quad + 2[2\delta(n-2) + \delta(n-3) + \delta(n-4)] \\
&\quad + [2\delta(n-3) + \delta(n-4) + \delta(n-5)] + [2\delta(n-4) + \delta(n-5) + \delta(n-6)] \\
&= 4\delta(n) + 6\delta(n-1) + 8\delta(n-2) + 6\delta(n-3) + 5\delta(n-4) + 2\delta(n-5) + \delta(n-6)
\end{aligned}
$$

Problem 3.27

Determine the stability for the following linear system.

$$
y(n) = 0.5x(n) + 100x(n-2) - 20x(n-10)
$$

solution

Stable system:

Consider the impulse response $h(n)$, having

$$
h(n) = 0.5\delta(n) + 100\delta(n-2) - 20\delta(n-10)
$$

Then, for y(n)

$$
\begin{aligned}
|y(n)| &= |\sum_{k=-\infty}^{\infty} h(k)x(n-k)| \\
&= |\sum_{k=-\infty}^{\infty} [0.5\delta(k) + 100\delta(k-2) - 20\delta(k-10)]x(n-k)| \\
&= |0.5x(n) + 100x(n-2) - 20x(n-10)| \\
&< M[|0.5| + |100| + |-20|] < 120.5M
\end{aligned}
$$

Hence, we obtain a bounded output with a bounded input.

Problem 3.28

Determine the stability for each of the following linear systems.

$$
\begin{aligned}
(a) \quad y(n) &= \sum_{k=0}^{\infty} 0.75^k x(n-k) \\
(b) \quad y(n) &= \sum_{k=0}^{\infty} 2^k x(n-k)
\end{aligned}
$$

solution

1. (a) Stable

   When $|x(n)| < M$, with the bounded input, we have

$$|y(n)| = \sum_{k=0}^{\infty} 0.75^k x(n-k)$$
$$< M \sum_{k=0}^{\infty} |0.75^k| = M \frac{1}{1-0.75} = 4M$$

Hence, we obtain a bounded output with a bounded input.

2. (b) Not stable

   When $|x(n)| < M$, with the bounded input, let us set $x(n) = u(n)$, M can be 2

$$y(n) = \sum_{k=\infty}^{\infty} 2^k u(k)u(n-k) = \sum_{k=\infty}^{\infty} 2^{n-k} u(n-k)u(k) = 2^n \sum_{k=0}^{n} 2^{-k} \quad [n \geq 0]$$
$$= 2^n \frac{1-2^{-n-1}}{1-0.5} \geq 2^n \quad [n \geq 0]$$

   So, we have

$$\lim_{n\to\infty} y(n) = \lim_{n\to\infty} 2^n = +\infty$$

   We will obtain the unbounded output.

Advanced Problems

Problem 3.30

Given each of the following discrete-time systems,

$$\begin{array}{ll} (b) & y(n) = x(n-1) + 0.5y(n-2) \\ (d) & y(n) = |x(n)| \end{array}$$

determine if the system is (1) linear or nonlinear; (2) time invariant or time varying; (3) causal or noncausal; and (4) stable or unstable.

solution

1. (b) Linear; (d) Nonlinear

   (b) Linear

   For any input $x_1(n), x_2(n)$, and $a, b \in R$

   set $x(n) \equiv ax_1(n) + bx_2(n)$

   we have

$$y_1(n) = \sum_{k=-\infty}^{\infty} h(k)x_1(n-k)$$

$$y_2(n) = \sum_{k=-\infty}^{\infty} h(k)x_2(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(k)[ax_1(n-k) + bx_2(n-k)]$$

$$= a[\sum_{k=-\infty}^{\infty} h(k)x_1(n-k)] + b[\sum_{k=-\infty}^{\infty} h(k)x_2(n-k)] = ay_1(n) + by_2(n)$$

(d) Nonlinear

Example: set $x_1(n) = -1, x_2(n) = 1, x(n) \equiv x_1(n) + x_2(n) = 0$

Having $y_1(n) = 1, y_2(n) = 1, y(n) = 0$

Here $y(n) \neq y_1(n) + y_2(n)$

2. (b) time invariant; (d) time invariant

(b) time invariant

For input $x_1(n)$, the shifted input of n samples $x_2(n) \equiv x_1(n-n_0)$

$$y_1(n) = \sum_{k=-\infty}^{\infty} h(k)x_1(n-k)$$

$$y_1(n-n_0) = \sum_{k=-\infty}^{\infty} h(k)x_1(n-n_0-k) \quad [\text{replacing n by } n-n_0]$$

$$y_2(n) = \sum_{k=-\infty}^{\infty} h(k)x_2(n-k) = \sum_{k=-\infty}^{\infty} h(k)x_1(n-n_0-k) \quad [\text{replacing } x_2(n-k) \text{ by } x_1(n-n_0-k)]$$

So, having

$$y_1(n-n_0) = y_2(n)$$

(d) time invariant

For input $x_1(n)$, the shifted input of n samples $x_2(n) \equiv x_1(n-n_0)$

$$y_1(n) = |x_1(n)|$$

$$y_1(n-n_0) = |x_1(n-n_0)| \quad [\text{replacing n by } n-n_0]$$

$$y_2(n) = |x_2(n)| = |x_1(n-n_0)| \quad [\text{replacing } x_2(n) \text{ by } x_1(n-n_0)]$$

So, having

$$y_1(n-n_0) = y_2(n)$$

3. (b) Causal; (d) Causal

(b) Since the output y(n) depends on the past input x(n-1) , and past output y(n-2), the system is causal.

(d) Since the output y(n) depends on the current input x(n) , the system is causal.

4. (b) Stable; (d) Stable

(b) When $|x(n)| < M$, with the bounded input, we have

$$h(n) - 0.5h(n-2) = \delta(n-1) \quad [h(-2) = h(-1) = 0]$$
$$h(n) = 0.5[(\frac{1}{\sqrt{2}})^{n-1} + (\frac{-1}{\sqrt{2}})^{n-1}]u(n-1)$$

Then, for y(n)

$$|y(n)| = |\sum_{k=-\infty}^{\infty} h(k)x(n-k)|$$
$$= |\sum_{k=-\infty}^{\infty} h(k+1)x(n-k-1)|$$
$$< M \sum_{k=-\infty}^{\infty} |h(k+1)|$$
$$< M \sum_{k=-\infty}^{\infty} |0.5[(\frac{1}{\sqrt{2}})^{k} + (\frac{-1}{\sqrt{2}})^{k}]u(k)|$$
$$< M \sum_{k=0}^{\infty} (\frac{1}{\sqrt{2}})^{k} = M\frac{1}{1-\frac{1}{\sqrt{2}}} = M\frac{\sqrt{2}}{\sqrt{2}-1}$$

Hence, we obtain a bounded output with a bounded input.

(d) When $|x(n)| < M$, with the bounded input, we have

$$|y(n)| = |x(n)| < M$$

Hence, we obtain a bounded output with a bounded input.

Problem 3.31

Given each of the following discrete-time systems,

$$(c) \quad y(n) = \text{round}[x(n)]$$

determine if the system is (1) linear or nonlinear; (2) time invariant or time varying; (3) causal or noncausal; and (4) stable or unstable.

solution

1. Nonlinear

   Example: set $x_1(n) = 3.4, x_2(n) = 0.2, x(n) \equiv x_1(n) + x_2(n) = 3.6$

   Having $y_1(n) = \text{round}[3.4] = 3, y_2(n) = \text{round}[0.2] = 0, y(n) = \text{round}[3.6] = 4$

   Here $y(n) \neq y_1(n) + y_2(n)$

2. time invariant

   For input $x_1(n)$, the shifted input of n samples $x_2(n) \equiv x_1(n-n_0)$

$$y_1(n) = \text{round}[x_1(n)]$$
$$y_1(n-n_0) = \text{round}[x_1(n-n_0)] \quad [\text{replacing n by } n-n_0]$$
$$y_2(n) = \text{round}[x_2(n)] = \text{round}[x_1(n-n_0)] \quad [\text{replacing } x_2(n) \text{ by } x_1(n-n_0)]$$

So, having

$$y_1(n-n_0) = y_2(n)$$

3. Causal

   Since the output y(n) depends on the current input x(n) , the system is causal.

4. Stable

   When $|x(n)| < M$, with the bounded input, we have

$$|y(n)| = |\,\text{round}[x(n)]| \le \max(|x(n)-0.5|, |x(n)+0.5|) \le |x(n)| + 0.5 < M + 0.5$$

Hence, we obtain a bounded output with a bounded input.

Problem 3.34

Given a relaxed discrete-time system,

$$y(n) - ay(n-1) = x(n)$$

1. Show that the impulse response is

$$h(n) = a^n u(n)$$

2. If the impulse response of a relaxed discrete-time system is found as

$$h(n) = \begin{cases} a^n & n \ge 0, n = \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

   determine the discrete-time system equation.

solution

1. If $a = 0$, we have $h(n) = h(n) - 0 \cdot h(n-1) = \delta(n) = 0^n u(n)$

   Now consider $a \neq 0$

   A. **Mathematical induction**

      1. **The initial case:** Here we need the additional condition: $h(-1) = 0$
      2. **The induction step:** Then from $h(n) - ah(n-1) = \delta(n)|_{n<0} = 0\ [n<0]$

         When $h(n) = 0\ [n<0]$, we can infer that

$$h(n) = 0\ [n<0] \Longrightarrow h(n-1) = \frac{h(n)-0}{a} = 0\ [n-1<0]$$

   Conclusion A: $h(n) = 0\ [n<0]$

B. **Mathematical induction**

1. **The initial case:** We know that $h(-1) = 0$

$h(0) - ah(-1) = \delta(n)|_{n=0} = 1$

So having $h(0) = 1 = a^0$

2. **The induction step:** Then from $h(n) - ah(n-1) = 0\ [n-1 \geq 0]$

When $h(n-1) = a^{n-1}\ [n-1 \geq 0]$, we can infer that

$$h(n-1) = a^{n-1}\ [n-1 \geq 0] \Longrightarrow h(n) = ah(n-1) + 0 = a^n\ [n \geq 0]$$

Conclusion B: $h(n) = a^n\ [n \geq 0]$

Combine conclusion A and B, we have $h(n) = a^n u(n)$ for $a \neq 0$

Summary: $h(n) = a^n u(n)$ for $\forall a \in R$

2. We know the impulse response $h(n)$, and we can write as

$$h(n) = 0.5[a^n + (-a)^n]u(n)$$

For $y(n), y(n-2)$, we have

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$
$$a^2 y(n-2) = \sum_{k=-\infty}^{\infty} a^2 h(k-2)x(n-k)$$

Because

$$\begin{aligned}
h(k) - a^2 h(k-2) &= 0.5[a^k + (-a)^k]u(k) - a^2 \cdot 0.5[a^{k-2} + (-a)^{k-2}]u(k-2) \\
&= 0.5[a^k + (-a)^k][u(k) - u(k-2)] \\
&= 0.5[a^k + (-a)^k][\delta(k) + \delta(k-1)] \\
&= 0.5[a^k + (-a)^k]\delta(k) + 0.5[a^k + (-a)^k]\delta(k-1) \\
&= 0.5[a^0 + (-a)^0]\delta(k) + 0.5[a^1 + (-a)^1]\delta(k-1) \\
&= \delta(k)
\end{aligned}$$

So, we have

$$\begin{aligned}
y(n) - a^2 y(n-2) &= \sum_{k=-\infty}^{\infty} [h(k) - a^2 h(k-2)]x(n-k) \\
&= \sum_{k=-\infty}^{\infty} \delta(k)x(n-k) \\
&= \sum_{k=-\infty}^{\infty} \delta(k)x(n-0) \\
&= x(n) \sum_{k=-\infty}^{\infty} \delta(k) \\
&= x(n)
\end{aligned}$$

Finally, the discrete-time system equation:

$$y(n) - a^2 y(n-2) = x(n)$$