Homework 1

Course Title: Digital Signal Processing I (Spring 2020)

Course Number: ECE53800

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Problem: Problems 2.2, 2.3, 2.7, 2.13, 2.14, 2.19, 2.20, 2.21, 2.22, 2.25, 2.27, 2.29, 2.30

2.36, 2.41, 2.42, 2.34 (MATLAB, speech.dat is loaded in BB)

Problems

Problem 2.2

Given an analog signal

$$x(t) = 5\cos(2\pi \cdot 2500t) + 2\cos(2\pi \cdot 3200t),\ \text{for } t \geq 0$$

sampled at a rate of 8,000 Hz,

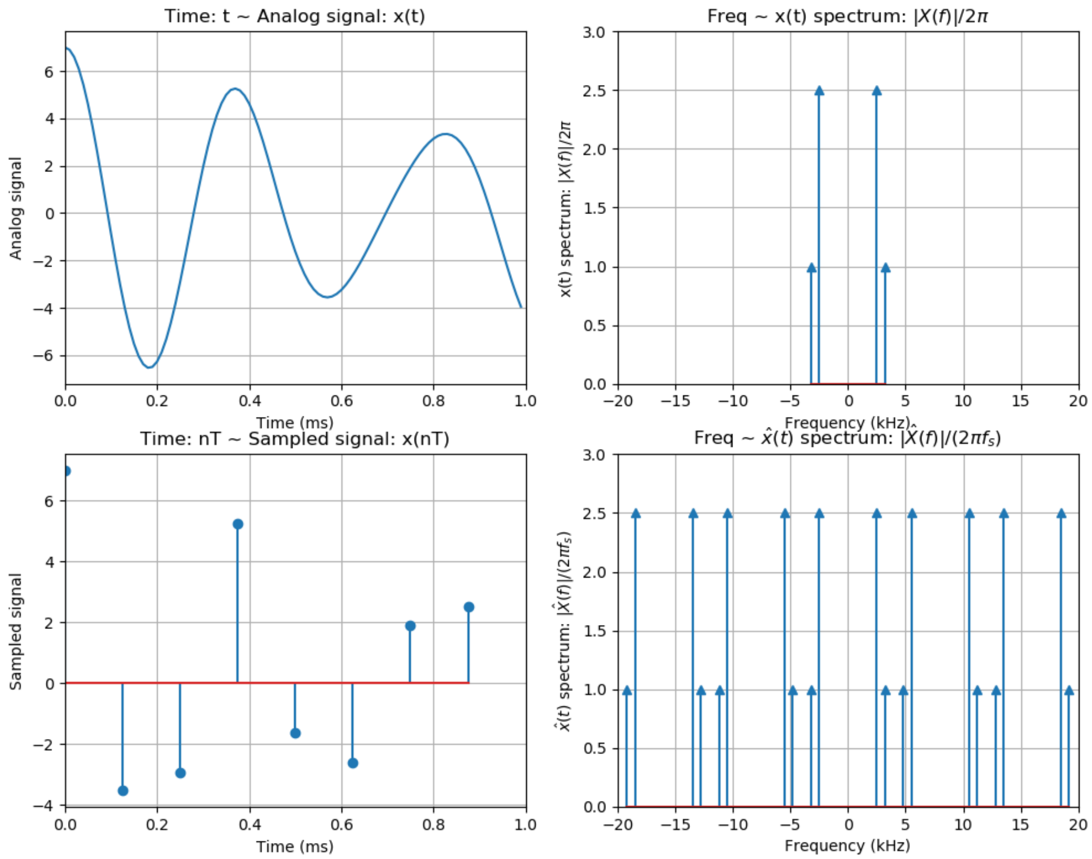
1. sketch the spectrum of the sampled signal up to 20 kHz;
2. sketch the recovered analog signal spectrum if an ideal lowpass filter with a cutoff frequency of 4 kHz is used to filter the sampled signal in order to recover the original signal.

solution

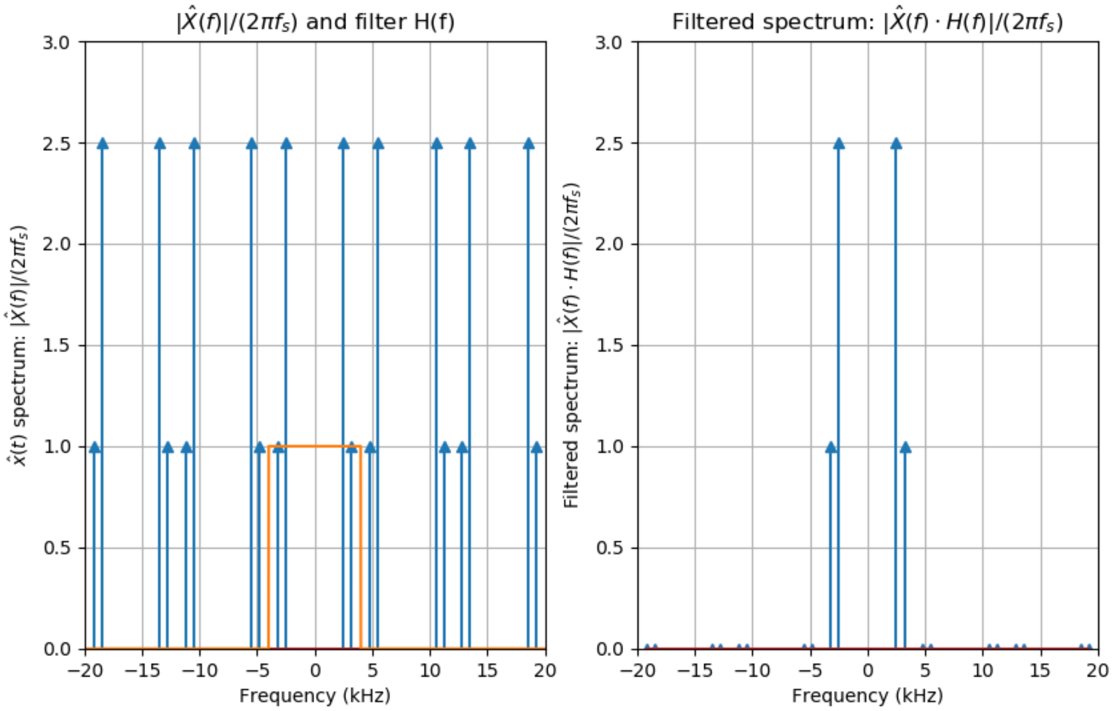
1. The frequency and spectrum of sampled signal up to 20 kHz are:

   freq: [-19.2, -18.5, -13.5, -12.8, -11.2, -10.5, -5.5, -4.8, -3.2, -2.5, 2.5, 3.2, 4.8, 5.5, 10.5, 11.2, 12.8, 13.5, 18.5, 19.2] kHz

   spectrum: [1, 2.5, 2.5, 1, 1, 2.5, 2.5, 1, 1, 2.5, 2.5, 1, 1, 2.5, 2.5, 1, 1, 2.5, 2.5, 1] $\times 2\pi f_s$



2. Filtered spectrum with a cutoff frequency of 4 kHz



Problem 2.3

Given an analog signal

$$x(t) = 3\cos(2\pi \cdot 1500t) + 2\cos(2\pi \cdot 2200t), \text{ for } t \geq 0$$

sampled at a rate of 8,000 Hz,

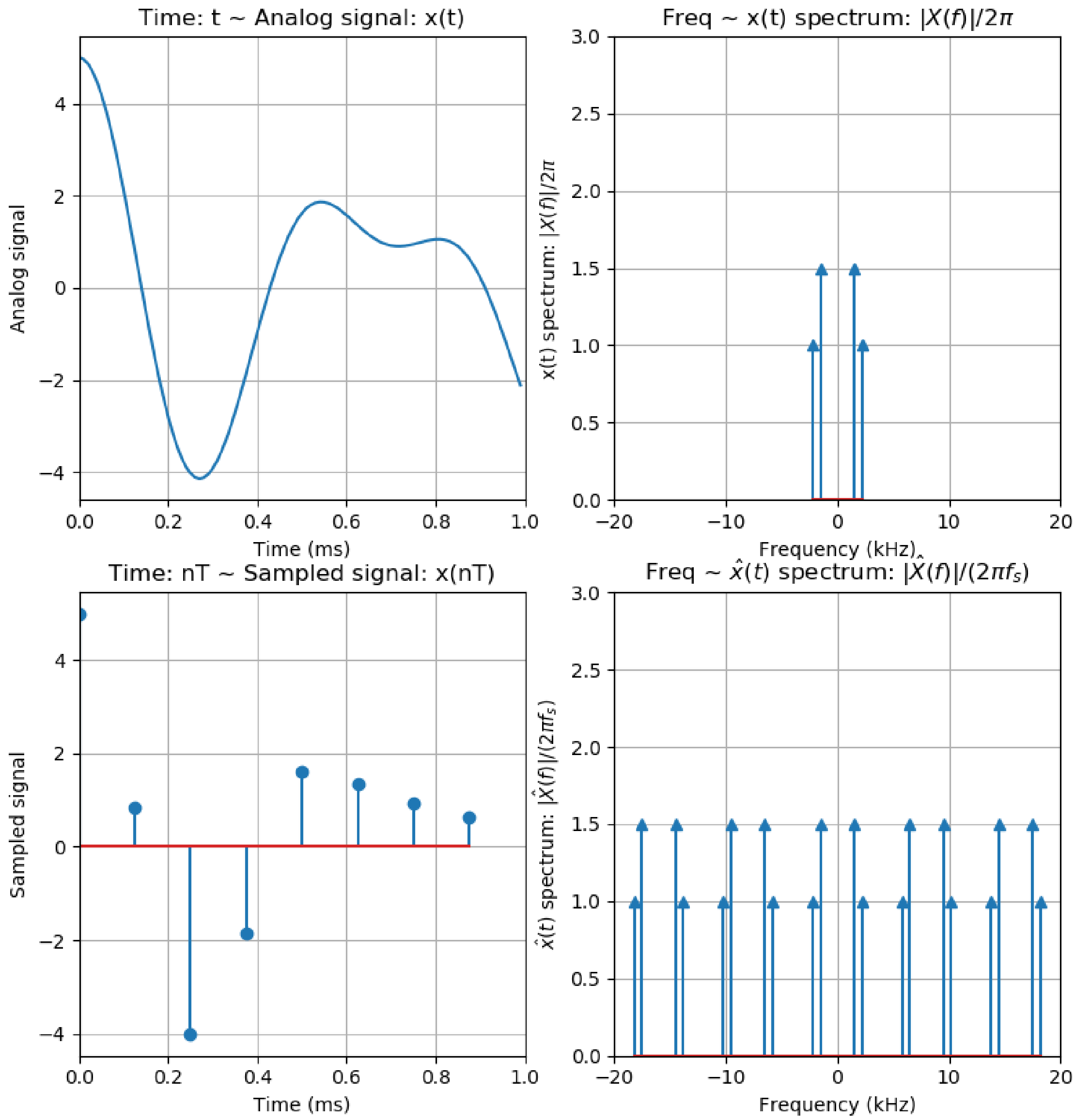
1. sketch the spectrum of the sampled signal up to 20 kHz;
2. sketch the recovered analog signal spectrum if an ideal lowpass filter with a cutoff frequency of 4 kHz is used to filter the sampled signal in order to recover the original signal.

solution

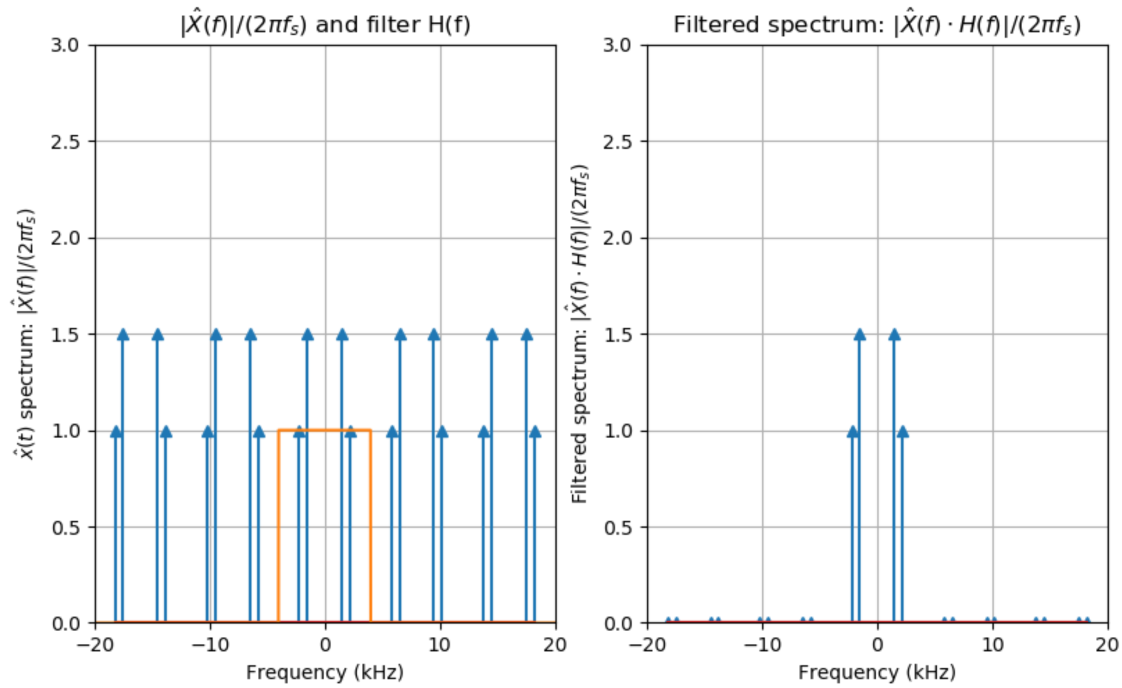
1. The frequency and spectrum of sampled signal up to 20 kHz are:

   freq: [-18.2, -17.5, -14.5, -13.8, -10.2, -9.5, -6.5, -5.8, -2.2, -1.5, 1.5, 2.2, 5.8, 6.5, 9.5, 10.2, 13.8, 14.5, 17.5, 18.2] kHz

   spectrum: [1, 1.5, 1.5, 1, 1, 1.5, 1.5, 1, 1, 1.5, 1.5, 1, 1, 1.5, 1.5, 1, 1, 1.5, 1.5, 1] $\times 2\pi f_s$



2. Filtered spectrum with a cutoff frequency of 4 kHz



Problem 2.7

Assuming a continuous signal is given as

$$x(t) = 8\cos(2\pi \cdot 5000t) + 5\sin(2\pi \cdot 7000t), \text{ for } t \geq 0$$

sampled at a rate of 8,000 Hz,

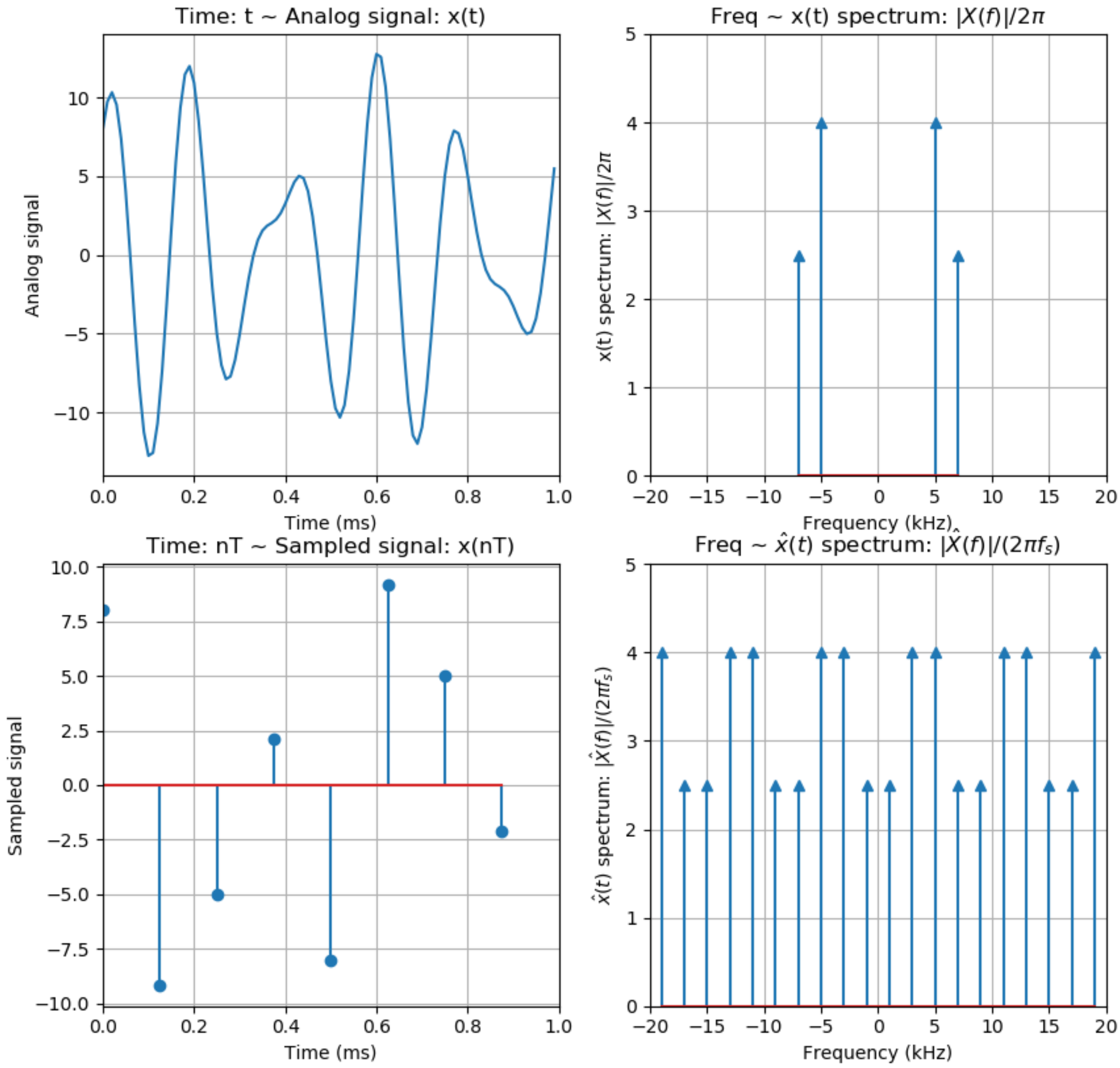
1. sketch the spectrum of the sampled signal up to 20 kHz;
2. sketch the recovered analog signal spectrum if an ideal lowpass filter with a cutoff frequency of 4 kHz is used to filter the sampled signal in order to recover the original signal;
3. determine the frequency/frequencies of aliasing noise.

solution

1. The frequency and spectrum of sampled signal up to 20 kHz are:

   freq: [-19.0, -17.0, -15.0, -13.0, -11.0, -9.0, -7.0, -5.0, -3.0, -1.0, 1.0, 3.0, 5.0, 7.0, 9.0, 11.0, 13.0, 15.0, 17.0, 19.0] kHz

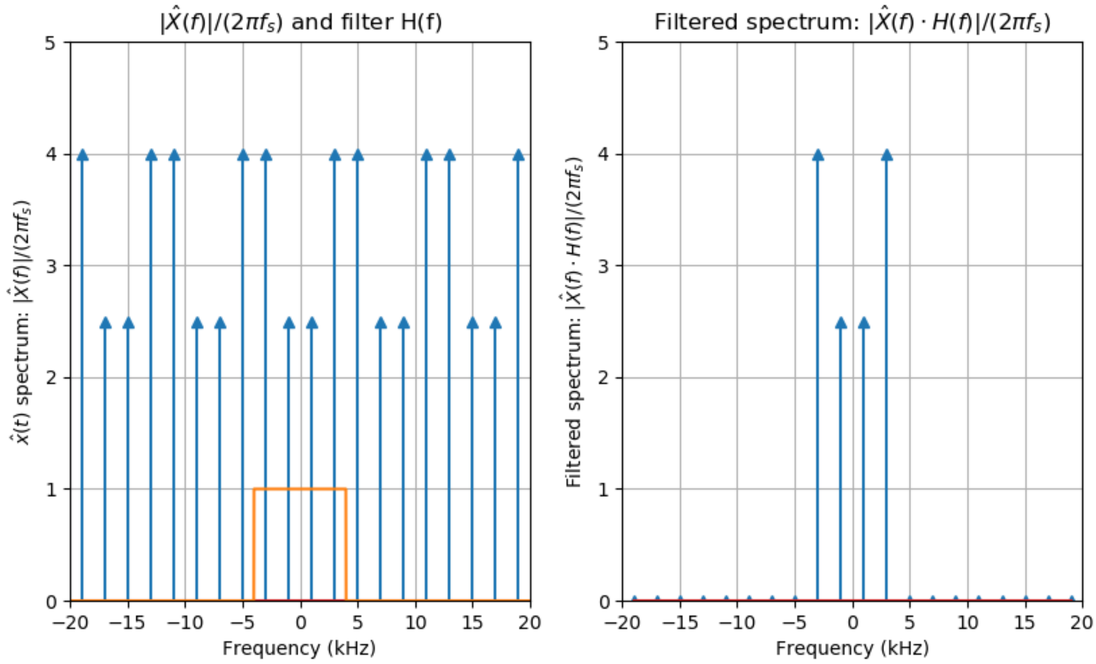
   spectrum: [4, -2.5j, 2.5j, 4, 4, -2.5j, 2.5j, 4, 4, -2.5j, 2.5j, 4, 4, -2.5j, 2.5j, 4, 4, -2.5j, 2.5j, 4] $\times 2\pi f_s$



2. Filtered spectrum with a cutoff frequency of 4 kHz

   freq: [-3.0, -1.0, 1.0, 3.0] kHz

   spectrum: [4, -2.5j, 2.5j, 4] $\times 2\pi$



3. Frequencies of aliasing noise:

   1 kHz, 3kHz

Problem 2.13

Given a DSP system in which a sampling rate of 8,000 Hz is used and the anti-aliasing filter is a second-order Butterworth lowpass filter with a cutoff frequency of 3.2 kHz, determine

1. the percentage of aliasing level at the cutoff frequency;
2. the percentage of aliasing level at the frequency of 1,000 Hz.

solution

We know, after sampling

$$\begin{aligned}\hat{x}(t) &= p(t)x(t) = [\sum \delta(t-nT)]x(t) \\ &= \sum x(nT)\delta(t-nT) \\ \hat{X}(f) &= \frac{1}{T}\sum X(f-k\frac{1}{T}) \\ &= f_s \sum X(f-kf_s)\end{aligned}$$

For specified frequency $0 < f_0 < f_s$, only consider the components $X(f), X(f-f_s)$ in $\hat{X}(f)$

Here we set Analog signal spectrum before anti-aliasing filter $X_{in}(f) = 1$, then

$$
\begin{aligned}
|X(f)|_{f=f_0} &= |X_{in}(f)\cdot H(f)|_{f=f_0} = H(f)|_{f=f_0} \\
|X(f-f_s)|_{f=f_0} &= |X_{in}(f-f_s)\cdot H(f-f_s)|_{f=f_0} = H(f-f_s)|_{f=f_0} \\
\text{aliasing level\%} &= \frac{|X(f-f_s)|_{f=f_0}}{|X(f)|_{f=f_0}} \\
&= \frac{H(f-f_s)|_{f=f_0}}{H(f)|_{f=f_0}} = \frac{H(f_0-f_s)}{H(f_0)} \\
&= \frac{\frac{1}{\sqrt{1+(\frac{f_0-f_s}{f_c})^{2n}}}}{\frac{1}{\sqrt{1+(\frac{f_0}{f_c})^{2n}}}} \\
&= \frac{\sqrt{1+(\frac{f_0}{f_c})^{2n}}}{\sqrt{1+(\frac{f_0-f_s}{f_c})^{2n}}}
\end{aligned}
$$

1. n=2, $f_0 = f_c = 3.2$ kHz

   $$\text{aliasing level\%} = \frac{\sqrt{1+(\frac{3.2}{3.2})^{2\times 2}}}{\sqrt{1+(\frac{3.2-8}{3.2})^{2\times 2}}} \times 100\%$$ = 57.44%

2. n=2, $f_0 = 1$ kHz

   $$\text{aliasing level\%} = \frac{\sqrt{1+(\frac{1}{3.2})^{2\times 2}}}{\sqrt{1+(\frac{1-8}{3.2})^{2\times 2}}} \times 100\%$$ = 20.55%

Problem 2.14

Given a DSP system in which a sampling rate of 8,000 Hz is used and the anti-aliasing filter is a Butterworth lowpass filter with a cutoff frequency 3.2 kHz, determine the order of the Butterworth lowpass filter for the percentage of aliasing level at the cutoff frequency required to be less than 10%.

solution

We know $f_c = 3.2$ kHz, f_s = 8 kHz

$$\text{aliasing level\%} = \frac{\sqrt{1+(\frac{f_0}{f_c})^{2n}}}{\sqrt{1+(\frac{f_0-f_s}{f_c})^{2n}}}|_{f_0=f_c} = \frac{\sqrt{2}}{\sqrt{1+(\frac{3.2-8}{3.2})^{2n}}} < 0.1 = 10\%$$

So, having

$$
\begin{aligned}
\frac{2}{0.1^2} - 1 < (1.5)^{2n} \\
\frac{1}{2}\frac{\ln(199)}{\ln(1.5)} = 6.53 < n
\end{aligned}
$$

In the end, we should choose $n \geq 7$ order of Butterworth lowpass filter

Problem 2.19

Given a DSP system with a sampling rate of 8,000 Hz and assuming that the hold circuit is used after DAC, determine

1. the percentage of distortion at the frequency of 3,000 Hz;
2. the percentage of distortion at the frequency of 1,600 Hz.

solution

We know that, $\delta(f) = 2\pi\delta(2\pi f) = 2\pi\delta(\omega)$, so

$$\begin{aligned}
y_H(t) &= y_s(t) * [-u(t-T) + u(t)] \\
Y_H(f) &= Y_s(f)H_h(f) = Y_s(f)[-e^{-j2\pi fT}(\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)) + (\frac{1}{j2\pi f} + \frac{1}{2}\delta(f))] \\
&= Y_s(f)\frac{-e^{-j2\pi fT}+1}{j2\pi f} \\
&= Y_s(f)\frac{e^{-j\pi fT}}{\pi f}\frac{e^{j\pi fT} - e^{-j\pi fT}}{j2} \\
&= Y_s(f)e^{-j\pi fT}\frac{\sin(\pi fT)}{\pi f} \\
&= Y_s(f)Te^{-j\pi fT}\frac{\sin(\pi fT)}{\pi fT} \\
&= \frac{Y_s(f)}{f_s}e^{-j\pi f/f_s}\frac{\sin(\pi f/f_s)}{(\pi f/f_s)}
\end{aligned}$$

Define distortion at f, here$(0 < f < f_s)$

$$\begin{aligned}
\text{distortion}\% &= \frac{|\frac{Y_s(f)}{f_s}| - |Y_H(f)|}{|\frac{Y_s(f)}{f_s}|} \\
&= 1 - |e^{-j\pi f/f_s}\frac{\sin(\pi f/f_s)}{(\pi f/f_s)}| \\
&= 1 - \frac{\sin(\pi f/f_s)}{(\pi f/f_s)}
\end{aligned}$$

1. f=3 kHz

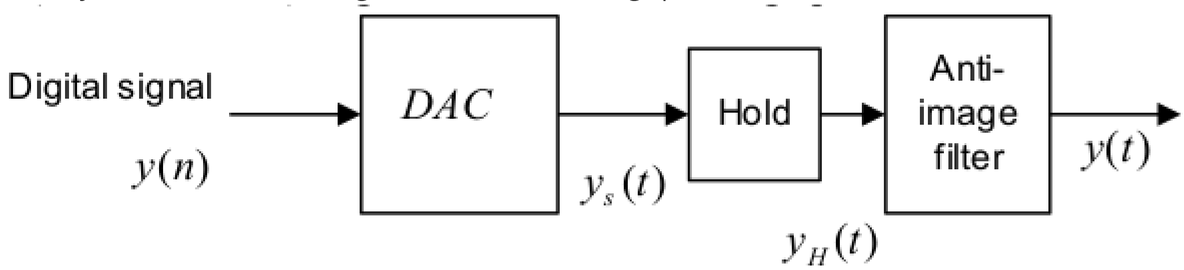
   $\text{distortion}\% = 1 - \frac{\sin(\pi\cdot 3/8)}{\pi\cdot 3/8}$ =21.58%

2. f=1.6 kHz

   $\text{distortion}\% = 1 - \frac{\sin(\pi\cdot 1.6/8)}{\pi\cdot 1.6/8}$ = 6.45%

Problem 2.20

A DSP system (FIGURE 2.40) is given with the following specifications:



Design requirements:

- Sampling rate 22,000 Hz
- Maximum allowable gain variation from 0 to 4,000 Hz = 2 dB
- 40 dB rejection at the frequency of 18,000 Hz
- Butterworth filter assumed

Determine the cutoff frequency and order for the anti-image filter.

solution

Define the cutoff frequency f_c, the order for the Butterworth filter n, we have

a. Gain variation from Hold block:

$$\begin{aligned}
\mathrm{G}_{Hold}|_{f=4kHz} &= 20\log(\frac{\sin(\pi f/f_s)}{(\pi f/f_s)})|_{f=4kHz} \\
&= 20\log(\frac{\sin(\pi 4/22)}{(\pi 4/22)}) \\
&= -0.477567\mathrm{dB} \\
\mathrm{G}_{Hold}|_{f=18kHz} &= 20\log(\frac{\sin(\pi f/f_s)}{(\pi f/f_s)})|_{f=18kHz} \\
&= 20\log(\frac{\sin(\pi 18/22)}{(\pi 18/22)}) \\
&= -13.541817\mathrm{dB}
\end{aligned}$$

b. Requirements for gain of Anti-image filter:

$$\begin{aligned}
\mathrm{G}_{Anti}|_{f=4kHz} &> -2\mathrm{dB} - \mathrm{G}_{Hold}|_{f=4kHz} \\
&= -1.522433\mathrm{dB} \\
\mathrm{G}_{Anti}|_{f=18kHz} &= -40\mathrm{dB} - \mathrm{G}_{Hold}|_{f=18kHz} \\
&= -26.458183\mathrm{dB}
\end{aligned}$$

c. In the other way, consider the Butterworth filter:

$$\begin{aligned}
\mathrm{G}_{Anti}|_{f=4kHz} &= 20\log(\frac{1}{\sqrt{1+(\frac{f}{f_c})^{2n}}})|_{f=4kHz} \\
&= -10\log(1+(\frac{f}{f_c})^{2n})|_{f=4kHz} \\
&= -10\log(1+(\frac{4}{f_c})^{2n}) \\
&> -1.522433\mathrm{dB} \\
\mathrm{G}_{Anti}|_{f=18kHz} &= 20\log(\frac{1}{\sqrt{1+(\frac{f}{f_c})^{2n}}})|_{f=18kHz} \\
&= -10\log(1+(\frac{f}{f_c})^{2n})|_{f=18kHz} \\
&= -10\log(1+(\frac{18}{f_c})^{2n}) \\
&= -26.458183\mathrm{dB}
\end{aligned}$$

d. Then, we conclude:

$$(\frac{4}{f_c})^{2n} < 10^{1.522433/10} - 1 = 0.419853$$
$$(\frac{18}{f_c})^{2n} = 10^{26.458183/10} - 1 = 441.403209$$

That is mean:

$$441.403209(\frac{4}{18})^{2n} = (\frac{18}{f_c})^{2n} \cdot (\frac{4}{18})^{2n} = (\frac{4}{f_c})^{2n} < 10^{1.522433/10} - 1 = 0.419853$$
$$1051.328703 = \frac{441.403209}{0.419853} < (\frac{18}{4})^{2n}$$
$$2.312983 = \frac{\log(1051.328703)}{2\log(\frac{18}{4})} < n$$

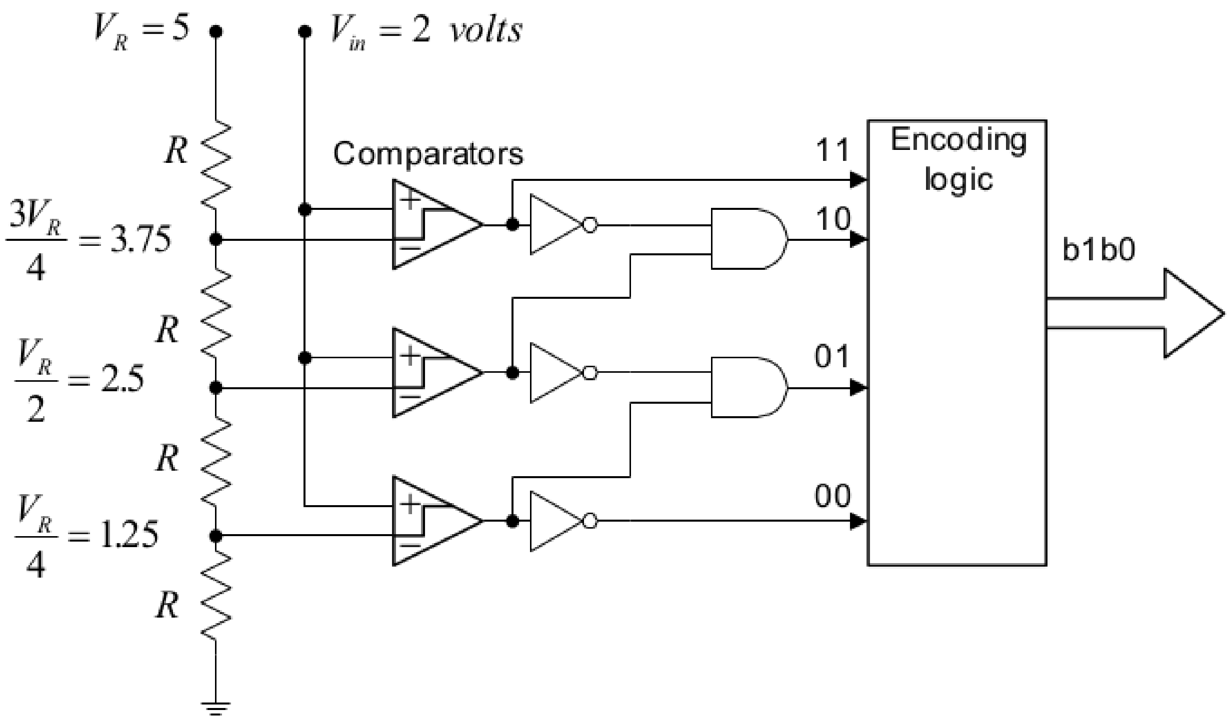
Now, set $n = 3 > 2.312983$, thus:

$$f_c = 18 \cdot (441.403209)^{\frac{-1}{2n}} = 18 \cdot (441.403209)^{\frac{-1}{6}} = 6.52329 \approx 6.52\text{kHz}$$

Finally, we can set: $f_c = 6.52\text{kHz}, n = 3$

Problem 2.21

Given the 2-bit flash ADC unit with an analog sample-and-hold voltage of 2 volts shown in Figure 2.41, determine the output bits.



solution

output of 3.75 V comparator: $A_2 = (2 > 3.75) = 0$

output of 2.5 V comparator: $A_1 = (2 > 2.5) = 0$

output of 1.25 V comparator: $A_0 = (2 > 1.25) = 1$

So, we have:

value on line 11: $A_2 = 0$

value on line 10: $\overline{A_2}A_1 = 1 \cdot 0 = 0$

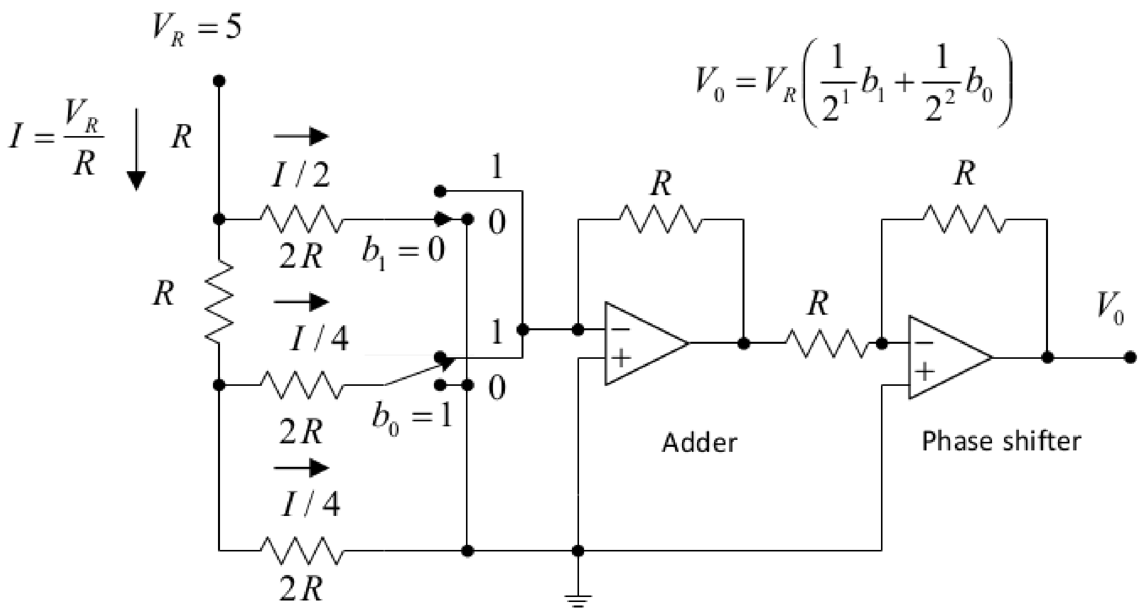
value on line 01: $\overline{A_1}A_0 = 1 \cdot 1 = 1$

value on line 00: $\overline{A_0} = 0$

Finally, the output bits $b_1 b_0$ are 01

Problem 2.22

Given the R-2R DAC unit with a 2-bit value as $b_1 b_0 = 01$ shown in Figure 2.42, determine the converted voltage.



solution

Consider general scenario, m bits$\overline{b_{n-1} \cdots b_1}$, define current on resistor of b_k bit as I_k, so

Overall current from V_R to Ground: I_{all}; Equivalent resistance from V_R to Ground: R_{all}

Current from R-2R network to Adder: I_{in} can be computed

$$
\begin{aligned}
R_{all} &= 2R//(R + 2R//\ldots) = 2R//(R + 2R//2R) = R \\
I_{all} &= \frac{V_R}{R_{all}} = \frac{V_R}{R} \\
I_k &= (\frac{1}{2})^m \cdot 2^k \cdot I_{all} \\
I_{in} &= \sum b_k \cdot I_k = (\frac{1}{2})^m \frac{V_R}{R} \sum b_k \cdot 2^k
\end{aligned}
$$

Then, for the voltage of Adder output: V_{adder}

the voltage of Phase shifter output: V_0; we have

$$0 - V_{adder} = I_{in} R$$
$$\frac{V_{adder} - 0}{R} = \frac{0 - V_0}{R}$$

Finally, we have

$$V_0 = -V_{adder} = I_{in} R = (\frac{1}{2})^m V_R \sum b_k \cdot 2^k$$

Here, m=2, $V_0 = V_R(\frac{1}{2^1} b_1 + \frac{1}{2^2} b_0)$, and $b_1 b_0 = 01, V_R = 5$,

$$V_0 = 5 \cdot (\frac{1}{2^1} 0 + \frac{1}{2^2} 1) = 1.25$$

Converted Voltage: V_0=1.25

Problem 2.25

Assuming that a 4-bit ADC channel accepts analog input ranging from 0 to 5 volts, determine the following:

1. number of quantization levels;
2. step size of quantizer or resolution;
3. quantization level when the analog voltage is 3.2 volts;
4. binary code produced by the ADC;
5. quantization error.

solution

1. number of quantization levels L is:

$$L = 2^m = 2^4 = 16$$

2. resolution Δ is:

$$\Delta = \frac{x_{max} - x_{min}}{L} = \frac{5 - 0}{16} = 0.3125\text{V}$$

3. quantization level i when the analog voltage is 3.2 volts:

$$i = \text{round}(\frac{x - x_{min}}{\Delta}) = \text{round}(\frac{3.2 - 0}{0.3125}) = 10$$

4. binary code $\overline{b_3 b_2 b_1 b_0}$ is:

$$\overline{b_3 b_2 b_1 b_0} = (10)_{10} = (1010)_2 = \overline{1010}$$

5. quantization error e_q is:

$$x_q = x_{min} + i \cdot \Delta = 0 + 10 \cdot 0.3125 = 3.125\text{V}$$
$$e_q \equiv x_q - x = 3.125 - 3.2 = -0.075\text{V}$$

Problem 2.27

Assuming that a 3-bit ADC channel accepts analog input ranging from -2.5 to 2.5 volts, determine the following:

1. number of quantization levels;
2. step size of quantizer or resolution;
3. quantization level when the analog voltage is -1.2 volts;
4. binary code produced by the ADC;
5. quantization error.

solution

1. number of quantization levels L is:

$$L = 2^m = 2^3 = 8$$

2. resolution Δ is:

$$\Delta = \frac{x_{max} - x_{min}}{L} = \frac{2.5 - (-2.5)}{8} = 0.625\text{V}$$

3. quantization level i when the analog voltage is -1.2 volts:

$$i = \text{round}(\frac{x - x_{min}}{\Delta}) = \text{round}(\frac{-1.2 - (-2.5)}{0.625}) = 2$$

4. binary code $\overline{b_2 b_1 b_0}$ is:

$$\overline{b_2 b_1 b_0} = (2)_{10} = (010)_2 = \overline{010}$$

5. quantization error e_q is:

$$x_q = x_{min} + i \cdot \Delta = -2.5 + 2 \cdot 0.625 = -1.25\text{V}$$
$$e_q \equiv x_q - x = -1.25 - (-1.2) = -0.05\text{V}$$

Problem 2.29

If the analog signal to be quantized is a sinusoidal waveform, that is,

$$x(t) = 9.5 \sin(2000 \times \pi t)$$

and if a bipolar quantizer uses 6 bits, determine

1. number of quantization levels;
2. quantization step size or resolution, Δ , assuming the signal range is from -10 to 10 volts;
3. the signal power to quantization noise power ratio

solution

1. number of quantization levels L is:

$$L = 2^m = 2^6 = 64$$

2. resolution, Δ , if the signal range is is from -10 to 10 volts:

$$\Delta = \frac{\text{Range} - (-\text{Range})}{L} = \frac{10 - (-10)}{L} = \frac{2 \cdot 10}{64} = 0.3125\text{V}$$

3. the signal power to quantization noise power ratio SNR_{dB}:

$$\begin{aligned}
\text{SNR}_{dB} &\equiv 10 \cdot \log(\frac{E\{x^2\}}{E\{e_q^2\}}) \\
&= 10 \cdot \log(\frac{A^2/2}{\frac{\Delta^2}{12}}) \\
&= 20 \cdot \log(\frac{A}{\Delta}) + 10 \log(6) \\
&= 20 \cdot \log(\frac{A}{\frac{2\text{Range}}{2^m}}) + 10 \log(6) \\
&= 20 \cdot \log(\frac{A}{\text{Range}}) + 20 \cdot \log(2) \cdot m + 10 \log(6/2^2) \\
&\approx 20 \cdot \log(\frac{A}{\text{Range}}) + 6.02 \cdot m + 1.76
\end{aligned}$$

Here, $A = 9.5, \text{Range} = 10, m = 6$, we have:

$$\text{SNR}_{dB} \approx 20 \cdot \log(\frac{9.5}{10}) + 6.02 \cdot 6 + 1.76 \approx 37.43\text{dB}$$

Problem 2.30

For a speech signal, if the ratio of the RMS value over the absolute maximum value of the signal is given, that is, $\left(\frac{x_{rms}}{|x|_{\max}}\right) = 0.25$ and the ADC bipolar quantizer uses 6 bits, determine

1. number of quantization levels;
2. quantization step size or resolution, Δ , if the signal range is 5 volts;
3. the signal power to quantization noise power ratio.

solution

1. number of quantization levels L is:

$$L = 2^m = 2^6 = 64$$

2. resolution, Δ , if the signal range is 5 volts:

$$\Delta = \frac{x_{max} - x_{min}}{L} = \frac{|x|_{max} - (-|x|_{max})}{L} = \frac{2 \cdot 5}{64} = 0.15625\text{V}$$

3. the signal power to quantization noise power ratio SNR_{dB}:

$$\begin{aligned}
\text{SNR}_{dB} &\equiv 10 \cdot \log(\frac{E\{x^2\}}{E\{e_q^2\}}) \\
&= 10 \cdot \log(\frac{x_{rms}^2}{\frac{\Delta^2}{12}}) \\
&= 20 \cdot \log(\frac{x_{rms}}{\Delta}) + 10 \cdot \log(12) \\
&= 20 \cdot \log(\frac{x_{rms}}{\frac{2|x|_{max}}{2^m}}) + 10 \cdot \log(12) \\
&= 20 \cdot \log(\frac{x_{rms}}{|x|_{max}}) + 20 \log(2) \cdot m + 10 \cdot \log(12/2^2) \\
&\approx 20 \cdot \log(\frac{x_{rms}}{|x|_{max}}) + 6.02 \cdot m + 4.77
\end{aligned}$$

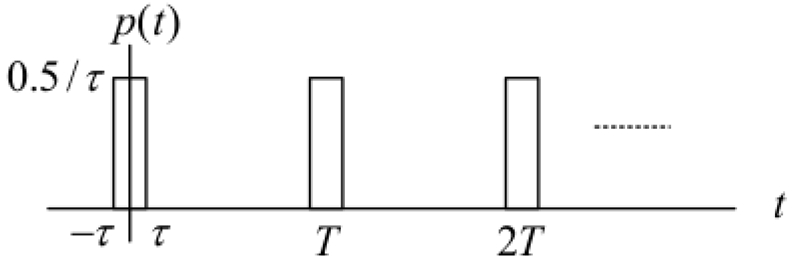
Here, $\frac{x_{rms}}{|x|_{max}} = 0.25, m = 6$, we have:

$$\text{SNR}_{dB} = 20 \cdot \log(0.25) + 6.02 \cdot 6 + 4.77 \approx 28.85\text{dB}$$

Advanced Problems

Problem 2.36

If the pulse train used is depicted in Figure 2.44,



1. determine the Fourier series expansion for $p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
2. determine $X_s(f)$ in terms of $X(f)$ using Fourier transform, that is,

$$X_s(f) = FT\{x_s(t)\} = FT\{x(t)p(t)\}$$

3. determine spectral distortion referring to $X(f)$ for $-f_s/2 < f < f_s/2$

solution

We know that $\omega_0 = 2\pi/T, 2\tau < T$, because $\int_T e^{-j(k-k')\frac{2\pi}{T}t}dt = 0$, when $k \neq k'$, so

$$\begin{aligned}
\int_T p(t)e^{-jk\frac{2\pi}{T}t}dt &= \int_T [\sum_{k'=-\infty}^{\infty} a_{k'} e^{jk'\frac{2\pi}{T}t}]e^{-jk\frac{2\pi}{T}t}dt \\
&= \int_T a_k e^{jk\frac{2\pi}{T}t} e^{-jk\frac{2\pi}{T}t}dt \\
&= Ta_k \\
a_k &= \frac{\int_T p(t)e^{-jk\frac{2\pi}{T}t}dt}{T} \\
&= \frac{0.5}{\tau T}\int_{-\tau}^{\tau} e^{-jk\frac{2\pi}{T}t}dt \\
&= \frac{0.5}{\tau T}\frac{e^{-jk\frac{2\pi}{T}t}|_{-\tau}^{\tau}}{-jk2\pi/T} \\
&= \frac{0.5}{\tau k\pi}\frac{e^{jk\frac{2\pi}{T}\tau} - e^{-jk\frac{2\pi}{T}\tau}}{j2} \\
&= \frac{0.5}{\tau k\pi}\sin(k\frac{2\pi}{T}\tau) \quad [k \neq 0] \\
a_0 &= \frac{0.5}{\tau T}\int_{-\tau}^{\tau} e^{-j0\frac{2\pi}{T}t}dt \\
&= \frac{1}{T}
\end{aligned}$$

1. the Fourier series expansion is

$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{T} + \sum_{k=-\infty(k\neq 0)}^{\infty} [\frac{0.5}{\tau k\pi}\sin(k\frac{2\pi}{T}\tau)]e^{jk\frac{2\pi}{T}t}$$

2. determine $X_s(f)$ in terms of $X(f)$, that is

$$\begin{aligned}
X_s(f) &= FT\{x_s(t)\} = FT\{x(t)p(t)\} \\
&= \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}[\frac{1}{T} + \sum_{k=-\infty(k\neq 0)}^{\infty} [\frac{0.5}{\tau k\pi}\sin(k\frac{2\pi}{T}\tau)]e^{jk\frac{2\pi}{T}t}]dt \\
&= \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft} + \sum_{k=-\infty(k\neq 0)}^{\infty} [\frac{0.5}{\tau k\pi}\sin(k\frac{2\pi}{T}\tau)]\int_{-\infty}^{+\infty} x(t)e^{-j2\pi(f-\frac{k}{T})t}dt \\
&= \frac{1}{T}X(f) + \sum_{k=-\infty(k\neq 0)}^{\infty} [\frac{0.5}{\tau k\pi}\sin(k\frac{2\pi}{T}\tau)]X(f-\frac{k}{T}) \\
&= f_s X(f) + \sum_{k=-\infty(k\neq 0)}^{\infty} [\frac{0.5}{\tau k\pi}\sin(k2\pi f_s\tau)]X(f-kf_s) \\
&= f_s X(f) + \sum_{k=-\infty(k\neq 0)}^{\infty} f_s[\frac{\sin(k2\pi f_s\tau)}{k2\pi f_s\tau}]X(f-kf_s)
\end{aligned}$$

3. For frequency f, ($-f_s/2 < f < f_s/2$), the distortion is

$$\begin{aligned}
\text{distortion}\% &= \frac{|f_s X(f)| - |X_s(f)|}{|f_s X(f)|} \\
&= 1 - |1 + \sum_{k=-\infty(k\neq 0)}^{\infty} [\frac{\sin(k2\pi f_s\tau)}{k2\pi f_s\tau}]\frac{X(f-kf_s)}{X(f)}|
\end{aligned}$$

If spectrum $X(f) = 0, (|f| > f_s/2)$,

Then $X(f-kf_s), (k\neq 0)$ has $X(f-kf_s) = 0$, for $-f_s/2 < f < f_s/2$, here

$$\text{distortion}\% = 1 - |1| = 0$$

Problem 2.41

Given the following modulated signal

$$x(t) = A_1\cos(\omega_1 t + \phi_1) \times A_2\cos(\omega_2 t + \phi_2)$$

with the signal ranging from $-A_1A_2$ to A_1A_2 determine the signal to quantization noise power ratio using m bits.

solution

Calculate $E\{x(t)^2\}$:

$$
\begin{aligned}
E\{x(t)^2\} &= \frac{A_1^2 A^2}{4} E\{2\cos^2(\omega_1 t + \phi_1) 2\cos^2(\omega_2 t + \phi_2)\} \\
&= \frac{A_1^2 A^2}{4} E\{[1 + \cos(2\omega_1 t + 2\phi_1)][(1 + \cos(2\omega_2 t + 2\phi_2)]\} \\
&= \frac{A_1^2 A^2}{4} E\{[1 + \cos(2\omega_1 t + 2\phi_1) + \cos(2\omega_2 t + 2\phi_2) + \cos(2\omega_1 t + 2\phi_1)\cos(2\omega_2 t + 2\phi_2)]\} \\
&= \frac{A_1^2 A^2}{4} E\{[1 + \cos(2\omega_1 t + 2\phi_1) + \cos(2\omega_2 t + 2\phi_2) + 0.5\cos(2(\omega_1 + \omega_2)t + 2(\phi_1 + \phi_2)) \\
&+ 0.5\cos(2(\omega_1 - \omega_2)t + 2(\phi_1 - \phi_2))]\} \\
&= \frac{A_1^2 A^2}{4} E\{1\} + 0 + 0 + 0 + 0 \\
&= \frac{A_1^2 A^2}{4}
\end{aligned}
$$

Suppose the distribution for Quantization Error is uniform:

$$
\Delta = \frac{A_1 A_2 - (-A_1 A_2)}{2^m} = \frac{2A_1 A_2}{2^m}
$$

$$
E\{e_q^2\} = \int_{-\Delta/2}^{\Delta/2} e_q^2 \frac{1}{\Delta} de_q = 2\frac{1}{3}(\frac{\Delta}{2})^3 \frac{1}{\Delta} = \frac{\Delta^2}{12}
$$

Calculate the the signal to quantization noise power ratio

$$
\begin{aligned}
\text{SNR}_{dB} &\equiv 10 \cdot \log(\frac{E\{x^2\}}{E\{e_q^2\}}) \\
&= 10 \cdot \log(\frac{\frac{A_1^2 A^2}{4}}{\frac{\Delta^2}{12}}) \\
&= 20 \cdot \log(\frac{A_1 A_2}{\Delta}) + 10\log(3) \\
&= 20 \cdot \log(\frac{A_1 A_2}{\frac{2A_1 A_2}{2^m}}) + 10\log(3) \\
&= 20 \cdot \log(2) \cdot (m - 1) + 10\log(3) \\
&= 20 \cdot \log(2) \cdot m + 10 \cdot \log(3/4) \\
&\approx [6.02 \cdot m - 1.25]\text{dB}
\end{aligned}
$$

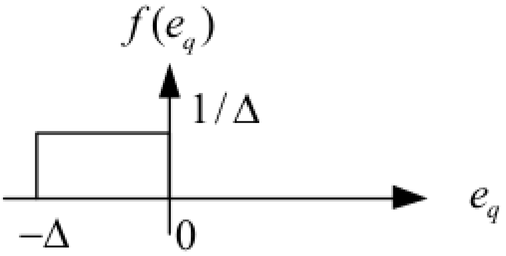
Finally, SNR is approximate $6.02m - 1.25$ dB

Problem 2.42

Assume that truncation of the continuous signal $x(n)$ in Problem 2.40 $x(t) = \sum_{i=1}^{N} A_i \cos(\omega_i t + \phi_i)$ranging from $-\sum_{i=1}^{N} A_i$ to $\sum_{i=1}^{N} A_i$ is defined below:

$$
x_q(n) = x(n) + e_q(n)
$$

where $-\Delta < e_q(n) \le 0$ and $\Delta = 2\sum_{i=1}^{N} A_i/2^m$. The quantized noise has the following distribution:



Determine the signal to quantization noise power ratio using m bits.

solution

Calculate $E\{x(t)^2\}$:

$$
\begin{aligned}
E\{x(t)^2\} &= E\{[\sum_{i=1}^{N} A_i \cos(\omega_i t + \phi_i)]^2\} \\
&= E\{[\sum_{i=1}^{N} A_i^2 \cos^2(\omega_i t + \phi_i) + 2\sum_{i>j} A_i A_j \cos(\omega_i t + \phi_i)\cos(\omega_j t + \phi_j)]\} \\
&= E\{[\sum_{i=1}^{N} \frac{1}{2} A_i^2 [1 + \cos(2\omega_i t + 2\phi_i)] \\
&+ \sum_{i>j} A_i A_j [\cos((\omega_i + \omega_j)t + (\phi_i + \phi_j)) + \cos((\omega_i - \omega_j)t + (\phi_i - \phi_j))]\} \\
&= E\{[\sum_{i=1}^{N} \frac{1}{2} A_i^2 [1 + \cos(2\omega_i t + 2\phi_i)]\} = E\{[\sum_{i=1}^{N} \frac{1}{2} A_i^2]\} \\
&= \frac{1}{2}\sum_{i=1}^{N} A_i^2
\end{aligned}
$$

Calculate $E\{e_q^2\}$:

$$
\Delta = \frac{2\sum_{i=1}^{N} A_i}{2^m}
$$

$$
E\{e_q^2\} = \int_{-\Delta}^{0} e_q^2 \frac{1}{\Delta} de_q = \frac{1}{3}\Delta^3 \frac{1}{\Delta} = \frac{\Delta^2}{3}
$$

Calculate the the signal to quantization noise power ratio:

$$
\begin{aligned}
\text{SNR}_{dB} &\equiv 10 \cdot \log(\frac{E\{x^2\}}{E\{e_q^2\}}) \\
&= 10 \cdot \log(\frac{\frac{1}{2}\sum_{i=1}^{N} A_i^2}{\frac{\Delta^2}{3}}) \\
&= 10 \cdot \log(\frac{\frac{1}{2}\sum_{i=1}^{N} A_i^2}{\frac{(\frac{2\sum_{i=1}^{N} A_i}{2^m})^2}{3}}) \\
&= 10 \cdot \log(\frac{3}{8} 2^{2m} \frac{\sum_{i=1}^{N} A_i^2}{(\sum_{i=1}^{N} A_i)^2}) \\
&= 10 \cdot \log(\frac{\sum_{i=1}^{N} A_i^2}{(\sum_{i=1}^{N} A_i)^2}) + 10\log(\frac{3}{8}) + 20\log(2) \cdot m \\
&\approx 10 \cdot \log(\frac{\sum_{i=1}^{N} A_i^2}{(\sum_{i=1}^{N} A_i)^2}) + 6.02 \cdot m - 4.26 \\
&= 10 \cdot \log(\sum_{i=1}^{N} A_i^2) - 20 \cdot \log(\sum_{i=1}^{N} A_i) + 6.02 \cdot m - 4.26
\end{aligned}
$$

Finally, we have SNR:

$$
\text{SNR}_{dB} \approx 10 \cdot \log(\sum_{i=1}^{N} A_i^2) - 20 \cdot \log(\sum_{i=1}^{N} A_i) + 6.02 \cdot m - 4.26
$$

MATLAB Projects

Problem 2.34

2.34. Performance evaluation of speech quantization:

Given an original speech segment “speech.dat” sampled at 8,000 Hz with each sample encoded in 16 bits,

1. use Programs 2.3-2.5 and modify Program 2.2 to quantize the speech segment using 3 to 15 bits, respectively.
2. The SNR in dB must be measured for each quantization.

[MATLAB function: “sound(x/max(abs(x)),fs)” can be used to evaluate sound quality, where “x” is the speech segment while “fs” is the sampling rate of 8,000 Hz.]

3. In this project, create a plot of the measured SNR (dB) versus the number of bits and discuss the effect of the sound quality.
4. For comparisons, plot the original speech and the quantized one using 3 bits, 8 bits, and 15 bits.

solution

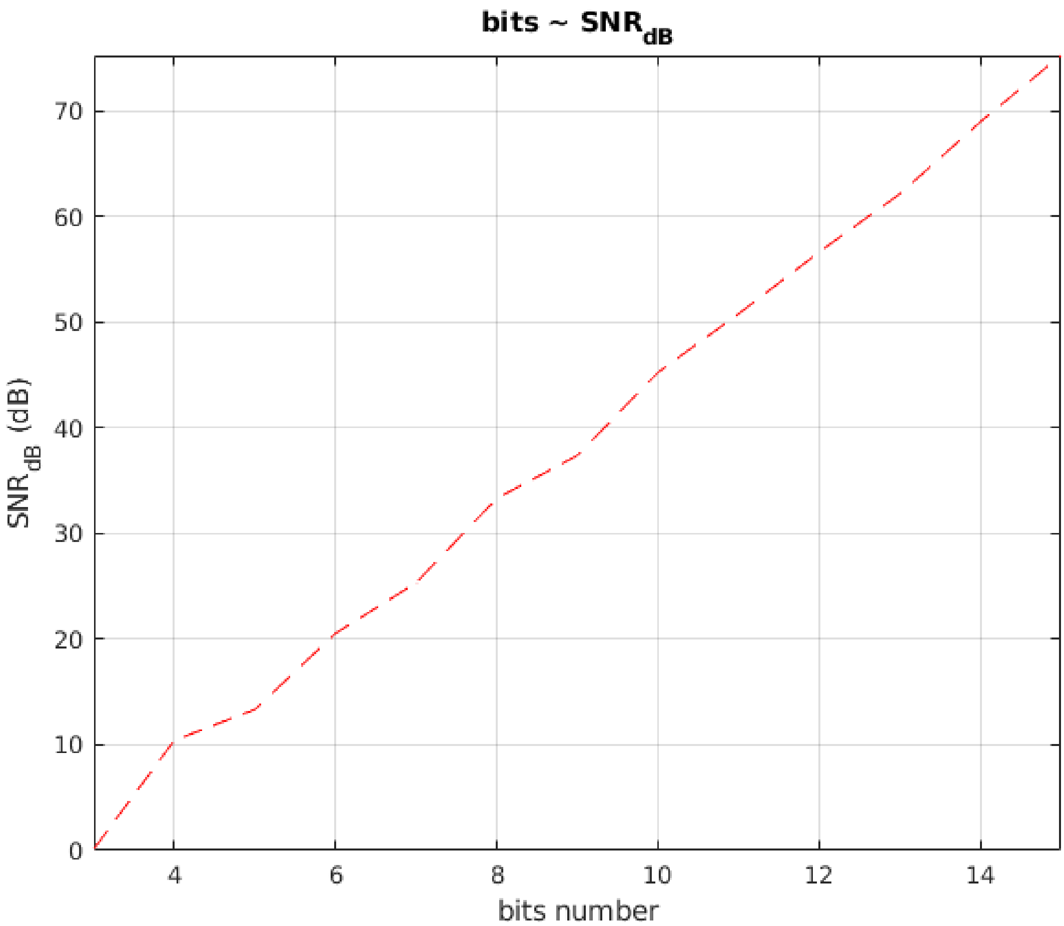
code: problem_2_34.m

```
% Author: Zhankun Luo
% Course Title: Digital Signal Processing I  (Spring 2020)
% Course Number: ECE53800
% Instructor: Dr. Li Tan
% Homework 1, problem 2.34
clear; clc; close all
load("speech.dat"); len = length(speech);
x_max = max(speech); x_min = min(speech);
fs = 8 * 1000; t = [1:1:len] * (1 / fs); % 8 kHz sampling freq
range_num_bits = [3, 15];
list_num_bits = range_num_bits(1):range_num_bits(end); % 3 bits ~ 15 bits
list_SNR = -1 * ones(1, range_num_bits(end) - range_num_bits(1)+1 );
for index = 1:length( list_num_bits )
    num_bits = list_num_bits(index);
    speech_q = zeros(1, len);
    for i = 1:len
        [~, speech_q(i)] = biquant(num_bits, x_min, x_max, speech(i) );
    end
    list_SNR(index) = snr(speech, speech_q);
end
figure(); plot(list_num_bits, list_SNR, "r--"); grid on
axis([range_num_bits(1), range_num_bits(end), 0, inf]); title("bits ~ SNR_{dB}")
xlabel("bits number"); ylabel("SNR_{dB} (dB)"); list_SNR
%% plot the original speech and the quantized one using 3, 8, 15 bits
list_plot = [3, 8, 15];
for index_plot = 1:length( list_plot )
    figure(); num_bits = list_plot(index_plot);
    speech_q = zeros(1, len);
    for i = 1:len
        [~, speech_q(i)] = biquant(num_bits, x_min, x_max, speech(i) );
    end
    err_q = speech_q - speech;
    subplot(3,1,1); plot(t, speech); grid on
    ylabel('Original speech'); title('speech.dat: "speech"');
    subplot(3,1,2); stairs(t, speech_q); grid on
    ylabel('Quantized speech')
    subplot(3,1,3); stairs(t, err_q); grid on
    ylabel('Quantized error'); xlabel('Time (s)'); ylim([x_min, x_max]);
    sound(speech_q/max(x_max, -x_min), fs); pause(2.5); % wait for 2.5 s
end
```

1. the speech segment is quantized from 3 to 15 bits with "biquant.m"
2. The SNR in dB is measured from 3 to 15 bits:

bits	SNR (dB)	bits	SNR (dB)
3	0.1545	10	45.2439
4	10.4267	11	50.7865
5	13.3577	12	56.6328
6	20.5624	13	62.1423
7	25.3478	14	68.9579
8	33.3398	15	75.2588
9	37.3870		

3. create a plot of the measured SNR (dB) versus the number of bits



discuss the effect of the sound quality:

We use "sound" function to hear the quantized speech segment with 3, 8, 15 bits respectively.

3 bits: sounds very Rough Voice

8 bits: much Smooth Voice

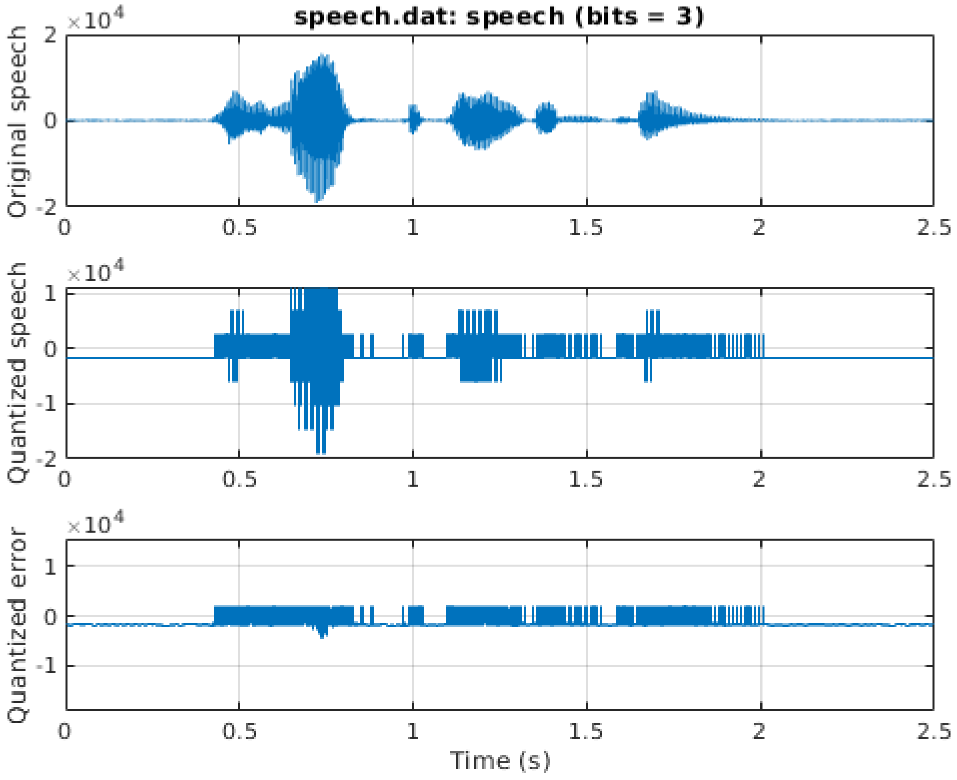
15 bits: ear can't not tell the difference between the quantized speed and original one

Conclusion:

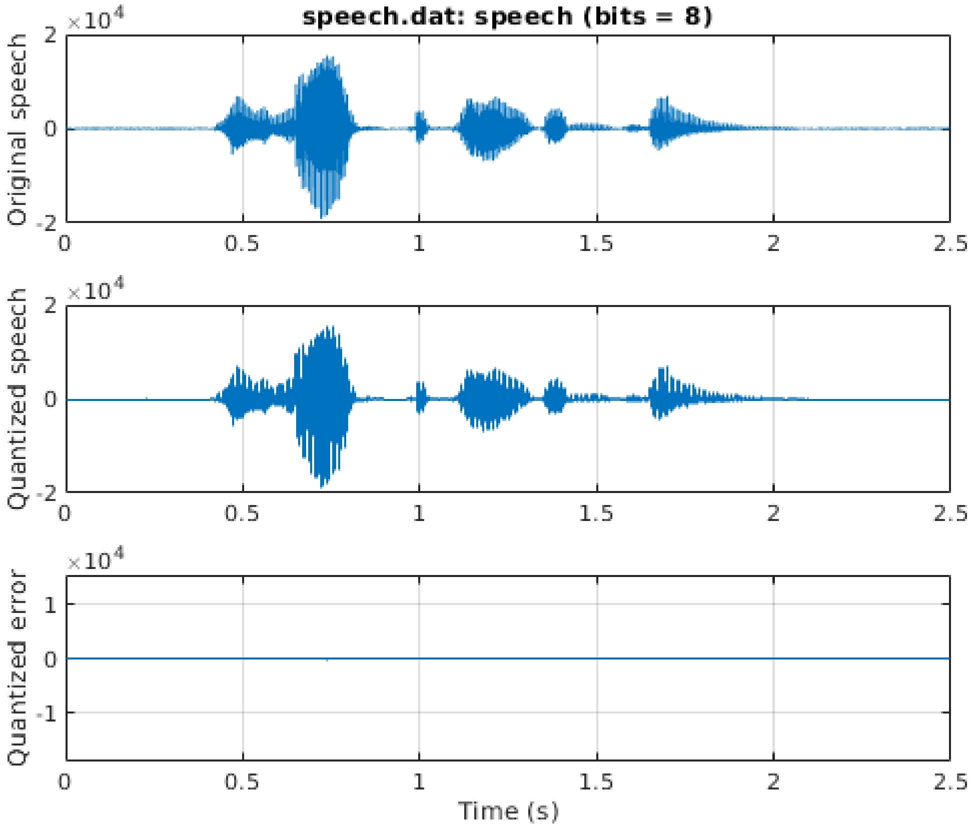
Increment of bits strongly improves the quality of the quantized speech segment.

4. For comparisons, plot the original speech and the quantized one using 3 bits, 8 bits, and 15 bits.

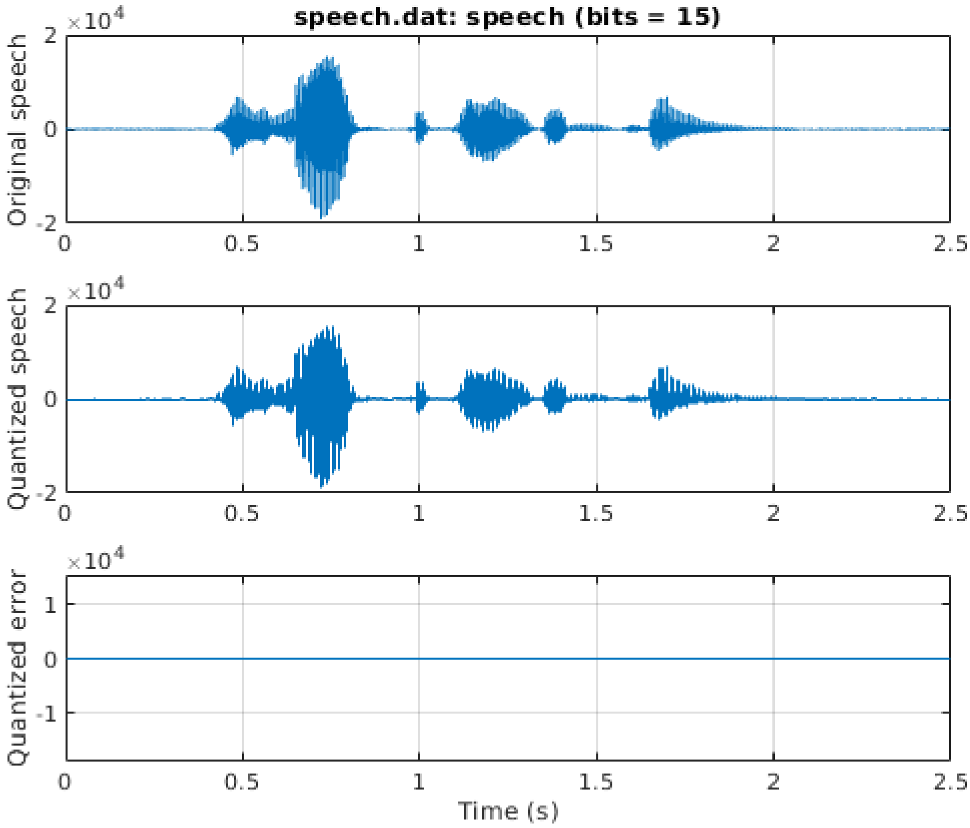
   a. 3 bits



   b. 8 bits


speech.dat: speech (bits = 8)
Original speech
Quantized speech
Quantized error
Time (s)

c. 15 bits


speech.dat: speech (bits = 15)
Original speech
Quantized speech
Quantized error
Time (s)