DTFT

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$
$$x^*(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

When $\hat{x}(t) = x(t)p(t) = x(t) \cdot \sum \delta(t-nT) = \sum x(nT)\delta(t-nT)$, then we have $\lambda < 1$ that

$$\begin{aligned}
\hat{X}(f) &= \int_{-\infty}^{\infty} \hat{x}(t)e^{-j2\pi ft}dt = \sum_n x(nT) \int_{-\infty}^{\infty} \delta(t-nT)e^{-j2\pi ft}dt \\
&= \sum_n x(nT)e^{-j2\pi n(f/f_s)} \\
\hat{x}^*(t) &= \int_{-\infty}^{\infty} \hat{X}(f)e^{j2\pi ft}df \\
x(nT) &= \int_{-\infty}^{\infty} \hat{x}^*(t)[\mathrm{u}(t-(n-\lambda)T) - \mathrm{u}(t-(n+\lambda)T)]dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{X}(f)e^{j2\pi ft}df[\mathrm{u}(t-(n-\lambda)T) - \mathrm{u}(t-(n+\lambda)T)]dt \\
&= \int_{-\infty}^{\infty} \hat{X}(f)df \int_{-\infty}^{\infty} e^{j2\pi ft}[\mathrm{u}(t-(n-\lambda)T) - \mathrm{u}(t-(n+\lambda)T)]dt \\
&= \int_{-\infty}^{\infty} \hat{X}(f)df \int_{(n-\lambda)T}^{(n+\lambda)T} e^{j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \hat{X}(f)df \int_{(n-\lambda)}^{(n+\lambda)} e^{j2\pi(f/f_s)t'}dt'/f_s \quad [t' = f_s t] \\
&= \int_{-\infty}^{\infty} \hat{X}(f)d(f/f_s)\frac{e^{j2\pi(f/f_s)(n+\lambda)} - e^{j2\pi(f/f_s)(n-\lambda)}}{j2\pi(f/f_s)} \\
&= \int_{-\infty}^{\infty} \hat{X}(f)d(f/f_s)e^{j2\pi(f/f_s)n}\frac{e^{j2\pi(f/f_s)\lambda} - e^{-j2\pi(f/f_s)\lambda}}{j2\pi(f/f_s)} \\
&= \int_{-\infty}^{\infty} \hat{X}(f)d(f/f_s)e^{j2\pi(f/f_s)n}\frac{2\lambda j\sin(2\pi(f/f_s)\lambda)}{j2\pi(f/f_s)\lambda} \\
&= 2\lambda\int_{-\infty}^{\infty} \hat{X}(f)d(f/f_s)e^{j2\pi(f/f_s)n}\frac{\sin(2\pi(f/f_s)\lambda)}{2\pi(f/f_s)\lambda}
\end{aligned}$$

If we replace kernel function: $[\mathrm{u}(t-(n-\lambda)T) - \mathrm{u}(t-(n+\lambda)T)]$ with function

$$\frac{\sin(\pi f_s(t-nT))}{\pi f_s(t-nT)} = \frac{e^{j2\pi(f_s/2)(t-nT)} - e^{-j2\pi(f_s/2)(t-nT)}}{j2\pi f_s(t-nT)}$$

We still have:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \hat{x}^*(t)\frac{\sin(\pi f_s(t-nT))}{\pi f_s(t-nT)}dt \\
&= \int_{-\infty}^{\infty} \sum_{n'} x(n'T)\delta(t-n'T)\frac{\sin(\pi f_s(t-nT))}{\pi f_s(t-nT)}dt \\
&= \sum_{n'} x(n'T)\int_{-\infty}^{\infty} \delta(t-n'T)\frac{\sin(\pi f_s(t-nT))}{\pi f_s(t-nT)}dt \\
&= \sum_{n'} x(n'T)\frac{\sin(\pi(f_s/f_s)(n'-n))}{\pi(f_s/f_s)(n'-n)}\int_{-\infty}^{\infty} \delta(t-n'T)dt \\
&= \sum_{n'} x(n'T)\delta(n'-n) = x(nT)
\end{aligned}$$

So, we have:

$$
\begin{aligned}
x(nT) &= \int_{-\infty}^{\infty} \hat{x}^*(t)\frac{\sin(\pi(t-nT))}{\pi(t-nT)}dt \\
&= \int_{-\infty}^{\infty} \hat{X}(f)df \int_{-\infty}^{\infty} e^{j2\pi ft}\frac{\sin(2\pi(f_s/2)(t-nT))}{2\pi(f_s/2)(t-nT)}dt \\
&= \int_{-\infty}^{\infty} \hat{X}(f)df \int_{-\infty}^{\infty} e^{j2\pi ft}\frac{e^{j2\pi(f_s/2)(t-nT)} - e^{-j2\pi(f_s/2)(t-nT)}}{2j2\pi(f_s/2)(t-nT)}dt \\
&= \int_{-\infty}^{\infty} \hat{X}(f)df\cdot[e^{-j2\pi n}\int_{-\infty}^{\infty} e^{j2\pi ft}\frac{e^{j2\pi(f_s/2)t}}{j2\pi f_s(t-nT)}dt - e^{j2\pi n}\int_{-\infty}^{\infty} e^{j2\pi ft}\frac{e^{-j2\pi(f_s/2)t}}{j2\pi f_s(t-nT)}dt] \\
&= \int_{-\infty}^{\infty} \hat{X}(f)df\cdot[e^{-j2\pi n}\int_{-\infty}^{\infty} e^{j2\pi(f/f_s)t'}\frac{e^{j\pi t'}}{j2\pi(t'-n)}dt'/f_s - e^{j2\pi n}\int_{-\infty}^{\infty} e^{j2\pi(f/f_s)t'}\frac{e^{-j\pi t'}}{j2\pi(t'-n)}dt'/f_s] \\
&= \int_{-\infty}^{\infty} \hat{X}(f)d(f/f_s)\cdot[e^{-j2\pi n}\int_{-\infty}^{\infty} e^{j2\pi(f/f_s+0.5)t'}\frac{1}{j2\pi(t'-n)}dt' - e^{j2\pi n}\int_{-\infty}^{\infty} e^{j2\pi(f/f_s-0.5)t'}\frac{1}{j2\pi(t'-n)} \\
&= \int_{-\infty}^{\infty} \hat{X}(f)d(f/f_s)\cdot[e^{-j2\pi n}\int_{-\infty}^{\infty} e^{j2\pi(f/f_s+0.5)(t'+n)}\frac{1}{j2\pi t'}dt' - e^{j2\pi n}\int_{-\infty}^{\infty} e^{j2\pi(f/f_s-0.5)(t'+n)}\frac{1}{j2\pi t'}dt'] \\
&= \int_{-\infty}^{\infty} \hat{X}(f)d(f/f_s)\cdot[e^{-j2\pi n}e^{j2\pi(f/f_s+0.5)n}\int_{-\infty}^{\infty} e^{j2\pi(f/f_s+0.5)t'}\frac{1}{j2\pi t'}dt' - e^{j2\pi n}e^{j2\pi(f/f_s-0.5)n}\int_{-\infty}^{\infty} e^{j2\pi(j} \\
&= \int_{-\infty}^{\infty} \hat{X}(f)d(f/f_s)[e^{j2\pi(f/f_s)n}\int_{-\infty}^{\infty} e^{j2\pi(f/f_s+0.5)t'}\frac{1}{j2\pi t'}dt' - e^{j2\pi(f/f_s)n}\int_{-\infty}^{\infty} e^{j2\pi(f/f_s-0.5)t'}\frac{1}{j2\pi t'}dt'] \\
&= \int_{-\infty}^{\infty} \hat{X}(f)d(f/f_s)e^{j2\pi(f/f_s)n}[\int_{-\infty}^{\infty} e^{j2\pi(f/f_s+0.5)t'}\frac{1}{j2\pi t'}dt' - \int_{-\infty}^{\infty} e^{j2\pi(f/f_s-0.5)t'}\frac{1}{j2\pi t'}dt'] \\
&= \int_{-\infty}^{\infty} \hat{X}(f)d(f/f_s)e^{j2\pi(f/f_s)n}\frac{1}{2}[\mathrm{sgn}(f/f_s+0.5) - \mathrm{sgn}(f/f_s-0.5)] \\
&= \int_{-\infty}^{\infty} \hat{X}(f)d(f/f_s)e^{j2\pi(f/f_s)n}[\mathrm{u}(f/f_s+0.5) - \mathrm{u}(f/f_s-0.5)] \\
&= \int_{-0.5}^{0.5} \hat{X}(f)e^{j2\pi(f/f_s)n}d(f/f_s)
\end{aligned}
$$

because $\mathrm{sgn}(t) \leftrightarrow \frac{1}{j\pi f}$, then we have $\frac{1}{j\pi t} \leftrightarrow \mathrm{sgn}(-f) = -\mathrm{sgn}(f)$,

Thus means

$$
\begin{aligned}
-\mathrm{sgn}(f) &= \int_{-\infty}^{\infty} \frac{1}{j\pi t}e^{-j2\pi ft}dt \\
\mathrm{sgn}(f) = -\mathrm{sgn}(-f) &= \int_{-\infty}^{\infty} \frac{1}{j\pi t}e^{j2\pi ft}dt \\
\mathrm{sgn}(f/f_s+0.5) &= \int_{-\infty}^{\infty} \frac{1}{j\pi t}e^{j2\pi(f/f_s+0.5)t}dt = 2\mathrm{u}(f/f_s+0.5) - 1 \\
\mathrm{sgn}(f/f_s-0.5) &= \int_{-\infty}^{\infty} \frac{1}{j\pi t}e^{j2\pi(f/f_s-0.5)t}dt = 2\mathrm{u}(f/f_s-0.5) - 1
\end{aligned}
$$

To sum up:

for $\hat{x}(t) = x(t)p(t) = x(t)\cdot\sum\delta(t-nT) = \sum x(nT)\delta(t-nT)$, we have

$$
\begin{aligned}
\hat{X}(f) &= \int_{-\infty}^{\infty} \hat{x}(t)e^{-j2\pi ft}dt \\
&= \sum_n x(nT)e^{-j2\pi n(f/f_s)} \\
\hat{x}^*(t) &= \int_{-\infty}^{\infty} \hat{X}(f)e^{j2\pi ft}df \\
x(nT) &= \int_{-\infty}^{\infty} \hat{x}^*(t)\frac{\sin(\pi(t-nT))}{\pi(t-nT)}dt \\
&= \int_{-0.5}^{0.5} \hat{X}(f)e^{j2\pi(f/f_s)n}d(f/f_s)
\end{aligned}
$$

Now define $\omega = 2\pi f/f_s$, then

$$X(e^{jw}) \equiv \hat{X}(f) = \sum_n x(nT)[e^{j\omega}]^{-n}$$

$$x(nT) = \int_{-0.5}^{0.5} \hat{X}(f) e^{j2\pi(f/f_s)n} d(f/f_s) = \frac{1}{2\pi} \int_{2\pi} X(e^{jw})[e^{j\omega}]^n d\omega$$

Formula for Comb function and Rectangular signal

$$\mathrm{FT}[\delta(t)] = \int \delta(t) e^{-j2\pi ft} dt = \int \delta(t) e^{-j2\pi f0} dt = 1 \cdot \int \delta(t) dt = 1(f)$$

$$\text{so, } \delta(t) = \mathrm{FT}^{-1}[1(f)] = \int 1 \cdot e^{j2\pi ft} df$$

then replace $f \to t, t \to -f$, having

$$\begin{aligned} \delta(-f) &= \int 1(t) \cdot e^{j2\pi t(-f)} dt = \mathrm{FT}[1(t)] \\ &= \delta(f) \end{aligned}$$

Thus Fourier pair $1(t) \leftrightarrow \delta(f)$, now we want to verify:

$$\begin{aligned} \sum_n \delta(t - nT) &= \frac{1}{T} \sum_k e^{j2\pi k(\frac{t}{T})} \quad [\alpha\delta(\alpha t') = \delta(t'), t' = \frac{t}{T} = tf_s] \\ \sum_n \delta(t' - n) &= \sum_k e^{j2\pi kt'} \end{aligned}$$

Here we notice: the Fourier transform (FT) of a rectangular pulse is the sinc function

The Fourier transform (DFT) of a rectangular signal after pulse discrete sampling is the Dirichlet function

$$\begin{aligned} \sum_{k=-A}^{A} e^{j2\pi kt'} &= 1 + 2\sum_{k=1}^{A} \cos(2\pi t' k) \\ \sin(\pi t') \sum_{k=-A}^{A} e^{j2\pi kt'} &= \sin(\pi t') + \sum_{k=1}^{A} [\sin(2\pi t'(k+0.5)) - \sin(2\pi t'(k-0.5))] \\ &= \sin(2\pi t'(A+0.5)) = \sin(\pi t'(2A+1)) \\ \sum_{k=-A}^{A} e^{j2\pi kt'} &= \frac{\sin(\pi t'(2A+1))}{\sin(\pi t')} \end{aligned}$$

Moreover, $\frac{\sin(\pi(t'+\Delta)(2A+1))}{\sin(\pi(t'+\Delta))} = \frac{\sin(\pi t'(2A+1))}{\sin(\pi t')}, \Delta \in Z$, the period is 1:

$$\int_{-0.5}^{0.5} \frac{\sin(\pi t'(2A+1))}{\sin(\pi t')} dt' = \sum_{k=-A}^{A} \int_{-0.5}^{0.5} e^{j2\pi kt'} dt' = \sum_{k=-A}^{A} \delta(k) = 1$$

So, $\lim_{A\to\infty} \frac{\sin(\pi t'(2A+1))}{\sin(\pi t')} [\mathrm{u}(t'+0.5) - \mathrm{u}(t'-0.5)] = \delta(t')$, then we have

$$\begin{aligned} \lim_{A\to\infty} \sum_{k=-A}^{A} e^{j2\pi kt'} &= \lim_{A\to\infty} \frac{\sin(\pi t'(2A+1))}{\sin(\pi t')} \\ &= \lim_{A\to\infty} \sum_n \frac{\sin(\pi t'(2A+1))}{\sin(\pi t')} [\mathrm{u}(t'+0.5-n) - \mathrm{u}(t'-0.5-n)] \\ &= \lim_{A\to\infty} \sum_n \frac{\sin(\pi(t'-n)(2A+1))}{\sin(\pi(t'-n))} [\mathrm{u}(t'+0.5-n) - \mathrm{u}(t'-0.5-n)] \\ &= \sum_n \lim_{A\to\infty} \frac{\sin(\pi(t'-n)(2A+1))}{\sin(\pi(t'-n))} [\mathrm{u}(t'+0.5-n) - \mathrm{u}(t'-0.5-n)] \\ &= \sum_n \delta(t'-n) \end{aligned}$$

Thus,

$$\begin{aligned} \sum_n \delta(t' - n) &= \sum_k e^{j2\pi kt'} \quad [t = t'T] \\ T\sum_n \delta(t - nT) = \sum_n \delta(t/T - n) &= \sum_k e^{j2\pi k(\frac{t}{T})} \end{aligned}$$

DFT

Sample $\hat{x}(t) = x(t)p(t) = x(t) \cdot \sum \delta(t - nT) = \sum x(nT)\delta(t - nT)$ in a period 0~NT

$$\begin{aligned}
\tilde{x}(t) &= \left\{\hat{x}(t)\left[\mathrm{u}(t) - \mathrm{u}(t - NT)\right]\right\} * \sum_{n'} \delta(t - n'NT) \\
&= \left\{\sum_{n=0}^{N-1} x(nT)\delta(t - nT)\right\} * \sum_{n'} \delta(t - n'NT) \\
&= \sum_{n'} \left\{\sum_{n=0}^{N-1} x(nT)\delta(t - (n + n'N)T)\right\}
\end{aligned}$$

For frequency domain:

$$\begin{aligned}
\tilde{X}(f) &= \int_{-\infty}^{\infty} \tilde{x}(t)e^{-j2\pi ft}dt \\
&= \sum_{n'} \left\{\sum_{n=0}^{N-1} x(nT)[\int_{-\infty}^{\infty} \delta(t - (n + n'N)T)e^{-j2\pi ft}dt]\right\} \\
&= \sum_{n'} \left\{\sum_{n=0}^{N-1} x(nT)e^{-j2\pi(f/f_s)(n+n'N)}\right\} \\
&= \sum_{n'} e^{-j2\pi(\frac{f}{f_s/N})n'} \left\{\sum_{n=0}^{N-1} x(nT)e^{-j2\pi(f/f_s)n}\right\} \quad [\sum_{n} \delta(t' - n) = \sum_{k} e^{j2\pi kt'}] \\
&= \sum_{k} \delta(\frac{f}{f_s/N} - k) \left\{\sum_{n=0}^{N-1} x(nT)e^{-j2\pi(f/f_s)n}\right\} \\
&= \frac{f_s}{N} \sum_{k} \delta(f - k\frac{f_s}{N}) \left\{\sum_{n=0}^{N-1} x(nT)e^{-j2\pi(f/f_s)n}\right\} \\
&= \frac{f_s}{N} \sum_{k} \delta(f - k\frac{f_s}{N}) \left\{\sum_{n=0}^{N-1} x(nT)e^{-j2\pi(k/N)n}\right\} \\
&= \sum_{k} \delta(f - k\frac{f_s}{N}) \left[\frac{f_s}{N} \left\{\sum_{n=0}^{N-1} x(nT)e^{-j2\pi(k/N)n}\right\}\right]
\end{aligned}$$

Reconstruction:

$$
\begin{aligned}
\tilde{x}^*(t) &= \int_{-\infty}^{\infty} \tilde{X}(f) e^{j2\pi ft} df \\
&= \int_{-\infty}^{\infty} e^{j2\pi ft} df \sum_k \delta(f - k\frac{f_s}{N}) \left[\frac{f_s}{N} \left\{ \sum_{n=0}^{N-1} x(nT) e^{-j2\pi(k/N)n} \right\} \right] \\
&= \sum_k \left[\frac{f_s}{N} \left\{ \sum_{n=0}^{N-1} x(nT) e^{-j2\pi(k/N)n} \right\} \right] \int_{-\infty}^{\infty} e^{j2\pi ft} \delta(f - k\frac{f_s}{N}) df \\
&= \sum_k \left[\frac{f_s}{N} \left\{ \sum_{n=0}^{N-1} x(nT) e^{-j2\pi(k/N)n} \right\} \right] e^{j2\pi(k/N) f_s t} \\
x(nT) &= \int_{-\infty}^{\infty} \tilde{x}^*(t) \frac{\sin(\pi(t - nT))}{\pi(t - nT)} dt \\
&= \int_{-0.5}^{0.5} \tilde{X}(f) e^{j2\pi(f/f_s)n} d(f/f_s) = \int_0^1 \tilde{X}(f) e^{j2\pi(f/f_s)n} d(f/f_s) \\
&= \int_0^1 e^{j2\pi(f/f_s)n} d(f/f_s) \sum_k \delta(f - k\frac{f_s}{N}) \left[\frac{f_s}{N} \left\{ \sum_{n=0}^{N-1} x(nT) e^{-j2\pi(k/N)n} \right\} \right] \\
&= \sum_k \int_0^1 e^{j2\pi(\frac{f}{f_s})n} \delta(\frac{f}{f_s} - \frac{k}{N}) d(\frac{f}{f_s}) \left[\frac{1}{N} \left\{ \sum_{n=0}^{N-1} x(nT) e^{-j2\pi(k/N)n} \right\} \right] \\
&= \sum_k \int_0^1 e^{j2\pi(\frac{k}{N})n} \delta(\frac{f}{f_s} - \frac{k}{N}) d(\frac{f}{f_s}) \left[\frac{1}{N} \left\{ \sum_{n=0}^{N-1} x(nT) e^{-j2\pi(k/N)n} \right\} \right] \\
&= \sum_{k=0}^{N-1} e^{j2\pi(\frac{k}{N})n} \left[\frac{1}{N} \left\{ \sum_{n=0}^{N-1} x(nT) e^{-j2\pi(k/N)n} \right\} \right] \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \sum_{n=0}^{N-1} x(nT) e^{-j2\pi(\frac{k}{N})n} \right\} e^{j2\pi(\frac{k}{N})n}
\end{aligned}
$$

To sum up, the Fourier transform (FT) of a rectangular pulse is the sinc function. The Fourier transform (FT) of a rectangular signal after pulse discrete sampling is the Dirichlet function. That is, DTFT is equivalent to FT first, then convolve with the sinc function; DFT is equivalent to FT first, then convolve with the Dirichlet function.