Formula of DCT

For a real sequence x(n) defined for $0 \leq n < N$, its discrete-cosine transform is given by

$$X_{\mathrm{DCT}}(k) = \sum_{n=0}^{N-1} 2x(n) \cos\left[\frac{\pi k(2n+1)}{2N}\right], 0 \leq k < N$$

Show that

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} a(k) X_{\mathrm{DCT}}(k) \cos\left[\frac{\pi k(2n+1)}{2N}\right], 0 \leq n < N$$

where

$$\alpha(k) = \begin{cases} 1/2 & k = 0 \\ 1 & 0 < k < N \end{cases}$$

solution (derive DCT from DFT)

Consider the new sequence:

$$x_{new}(n) = [x(0) \cdots x(N-1), x(N-1), \cdots x(0)]$$

Then, we find the relationship between 2N point DFT and N point DCT:

$$
\begin{aligned}
X_{new}(k) &= \text{DFT}[x_{new}(n)] = \sum_{n=0}^{N-1} x(n)e^{-j2\pi\frac{kn}{2N}} + \sum_{n=N}^{2N-1} x(-n+2N-1)e^{-j2\pi\frac{kn}{2N}} \\
&= \sum_{n=0}^{N-1} x(n)e^{-j2\pi\frac{kn}{2N}} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi\frac{k(-n-1+2N)}{2N}} \\
&= \sum_{n=0}^{N-1} x(n)[e^{-j2\pi\frac{kn}{2N}} + e^{j2\pi\frac{k(n+1)}{2N}}] \\
&= e^{j2\pi\frac{k}{4N}} \sum_{n=0}^{N-1} 2x(n)\cos\left[\frac{\pi k(2n+1)}{2N}\right] \\
&= \text{Phase} \times \text{Amplitude} \\
&= e^{j2\pi\frac{k}{4N}} X_{DCT}(k) \\
X_{new}(2N-k) &= e^{j2\pi\frac{(2N-k)}{4N}} \sum_{n=0}^{N-1} 2x(n)\cos\left[\frac{\pi(2N-k)(2n+1)}{2N}\right] \\
&= (-1)e^{-j2\pi\frac{k}{4N}} \sum_{n=0}^{N-1} 2x(n)\cos\left[\frac{\pi(-2N+k)(2n+1)}{2N}\right] \\
&= (-1)e^{-j2\pi\frac{k}{4N}} \sum_{n=0}^{N-1} 2x(n)\cos\left[\frac{\pi k(2n+1)}{2N} - \pi(2n+1)\right] \\
&= (-1)^{2n+2}e^{-j2\pi\frac{k}{4N}} \sum_{n=0}^{N-1} 2x(n)\cos\left[\frac{\pi k(2n+1)}{2N}\right] \\
&= e^{-j2\pi\frac{k}{4N}} \sum_{n=0}^{N-1} 2x(n)\cos\left[\frac{\pi k(2n+1)}{2N}\right] \\
&= e^{-j2\pi\frac{k}{4N}} X_{DCT}(k) \\
X_{new}(N) &= e^{j2\pi\frac{N}{4N}} \sum_{n=0}^{N-1} 2x(n)\cos\left[\frac{\pi N(2n+1)}{2N}\right] \\
&= e^{j2\pi\frac{1}{4}} \sum_{n=0}^{N-1} 2x(n)\cos\left[\frac{\pi}{2} + n\pi\right] = 0
\end{aligned}
$$

Notice that:

$$
\begin{aligned}
[e^{-j2\pi\frac{kn}{2N}} + e^{j2\pi\frac{k(n+1)}{2N}}] &= e^{j2\pi\frac{k}{4N}}[e^{-j2\pi\frac{k(n+\frac{1}{2})}{2N}} + e^{j2\pi\frac{k(n+\frac{1}{2})}{2N}}] \\
&= e^{j2\pi\frac{k}{4N}} 2\cos[2\pi\frac{k(n+\frac{1}{2})}{2N}] \\
&= e^{j2\pi\frac{k}{4N}} 2\cos\left[\frac{\pi k(2n+1)}{2N}\right]
\end{aligned}
$$

So, we know that from the inverse of DFT:

$$
\begin{aligned}
2Nx_{new}(n) = 2N \times \mathrm{IDFT}[X_{new}(k)] &= \sum_{k=0}^{N-1} X_{new}(k) e^{j2\pi\frac{kn}{2N}} + 0 + \sum_{k=N+1}^{2N-1} X_{new}(k) e^{j2\pi\frac{kn}{2N}} \\
&= \sum_{k=0}^{N-1} X_{new}(k) e^{j2\pi\frac{kn}{2N}} + \sum_{k=1}^{N-1} X_{new}(2N-k) e^{j2\pi\frac{(2N-k)n}{2N}} \\
&= X_{new}(0) + \sum_{k=1}^{N-1} [X_{new}(k) e^{j2\pi\frac{kn}{2N}} + X_{new}(2N-k) e^{j2\pi\frac{(2N-k)n}{2N}}] \\
&= X_{DCT}(0) + \sum_{k=1}^{N-1} [e^{j2\pi\frac{k}{4N}} X_{DCT}(k) e^{j2\pi\frac{kn}{2N}} + e^{-j2\pi\frac{k}{4N}} X_{DCT}(k) e^{-j2\pi\frac{kn}{2N}}] \\
&= X_{DCT}(0) + \sum_{k=1}^{N-1} X_{DCT}(k) [e^{j2\pi(\frac{k}{4N}+\frac{kn}{2N})} + e^{-j2\pi(\frac{k}{4N}+\frac{kn}{2N})}] \\
&= X_{DCT}(0) + \sum_{k=1}^{N-1} X_{DCT}(k) 2\cos\left[\frac{\pi k(2n+1)}{2N}\right]
\end{aligned}
$$

Thus, we conclude:

$$
\begin{aligned}
x(n) = x_{new}(n) &= \frac{1}{N}[\frac{1}{2} X_{DCT}(0) + \sum_{k=1}^{N-1} X_{DCT}(k)\cos\left[\frac{\pi k(2n+1)}{2N}\right]] \\
&= \frac{1}{N}\sum_{k=0}^{N-1} a(k) X_{\mathrm{DCT}}(k)\cos\left[\frac{\pi k(2n+1)}{2N}\right]
\end{aligned}
$$

where

$$
\alpha(k) = \begin{cases} 1/2 & k = 0 \\ 1 & 0 < k < N \end{cases}
$$

Comment:

$X_{\mathrm{DCT}}(k)$ actually stores the "amplitude" information of DFT, thus we don't need to store the "phase". It costs less memory to store the complete information.