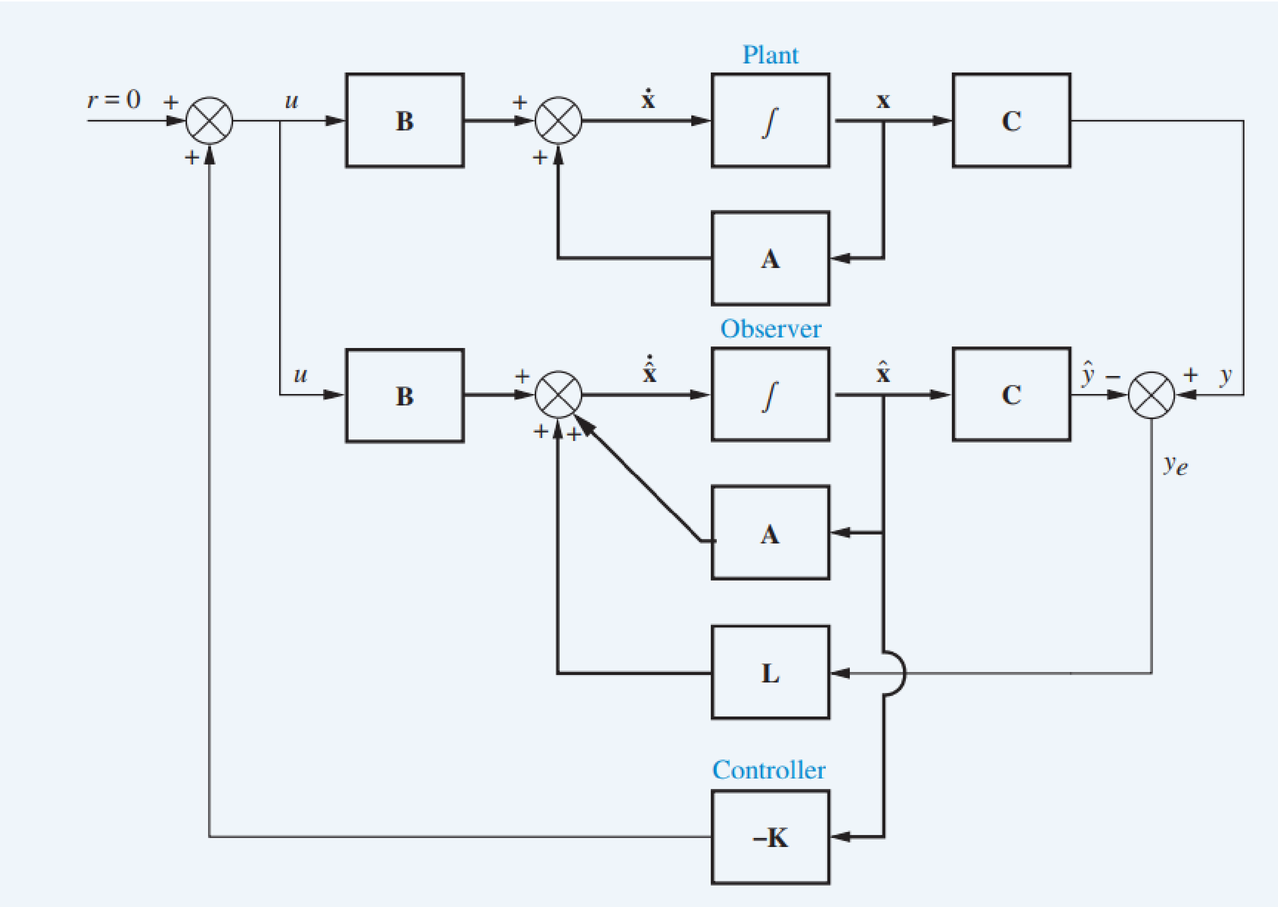
Design Via State Space



Controllability

The basic equation set: (D alaways = 0)

$$\dot{X} = AX + BU$$
$$Y = CX + D$$

Introduce the Controller K always 1xN, \
where U always 1x1

$$U = r - KX$$

So, we obtain

$$\dot{X} = AX + B(r - KU) = (A - BK)X + Br$$

if we could manipulate th poles of $|sI-(A-BK)|$
Thus means

$$Y = C\mathbb{L}^{-\mathbb{1}}[(sI-(A-BK))^{-1}] * \mathbb{L}^{-\mathbb{1}}[BR(s)]$$

Transformation

Here
$Z = PX$
We have

$$\dot{Z} = AZ + BU$$
$$Y = CZ$$
$$U = r - KZ$$

Thus

$$\dot{X} = P^{-1}APX + P^{-1}BU$$
$$Y = CPX$$
$$U = r - KPX$$

Compare with

$$\dot{X} = A_x X + B_x U$$
$$Y = C_x X$$
$$U = r - K_x X$$

So we obtain

$$A_x = P^{-1}AP$$
$$B_x = P^{-1}B$$
$$C_x = CP$$
$$K_x = KP$$

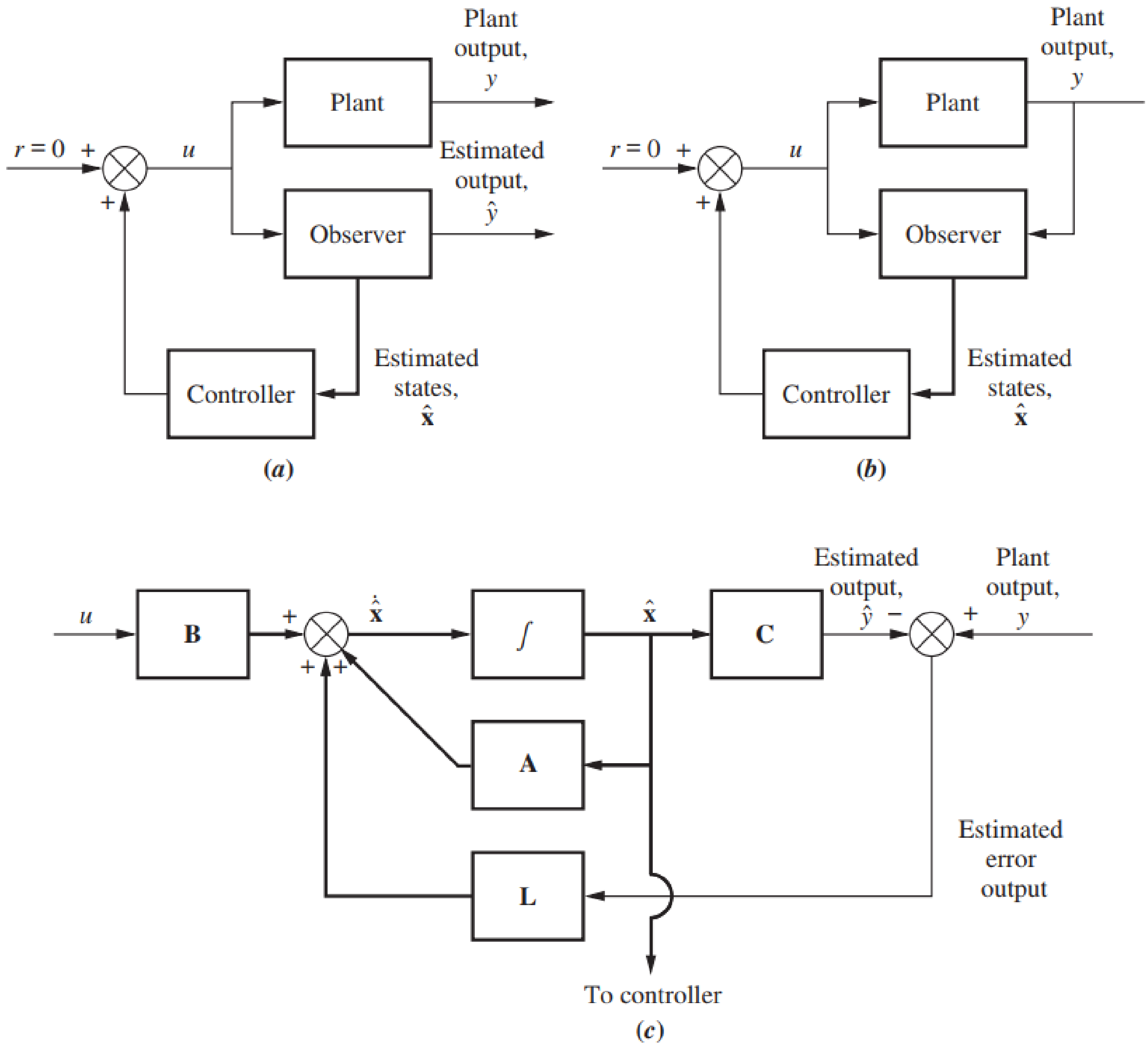
Then we have

$$C_{Mx} = [B_x \ A_x B_x \cdots A_x^{N-1} B_x] = P^{-1} C_{Mz}$$

where X is observer canonical form\
Z is other form (like phase variable form, cascade form)

$$P = C_{Mz} C_{Mx}^{-1} \ K_z = K_x P^{-1}$$

Observability



$$\dot{\hat{X}} = A\hat{X} + BU + L(Y - \hat{Y})$$
$$\hat{Y} = C\hat{X}$$

so with

$$\dot{X} = AX + BU$$
$$Y = CX$$

then obtain

$$\dot{X} - \dot{\hat{X}} = A(X - \hat{X}) - LC(X - \hat{X})$$
$$= (A - LC)(X - \hat{X})$$

define $e_X \equiv (X - \hat{X})$, we have

$$\dot{e}_X = (A - LC)e_X$$

If all poles of (A-LC) in the left plane

$$\lim_{t \to \infty} e_X = (X - \hat{X}) = 0$$

Then we could use $\hat{X}$ to estimate X\
regardless the influence of initial value $\hat{X}(0)$ and $X(0)$

Transformation

$Z = PX$
where X is observer canonical form\
Z is other form (like phase variable form, cascade form)

$$(\dot{Z} - \dot{\hat{Z}}) = (A - LC)(Z - \hat{Z})$$

Then we have

$$(\dot{X} - \dot{\hat{X}}) = P^{-1}(A - LC)P(X - \hat{X})$$

So we have:

$$\begin{aligned} A_x &= P^{-1}AP \\ B_x &= P^{-1}B \\ C_x &= CP \\ L_x &= P^{-1}L \end{aligned}$$

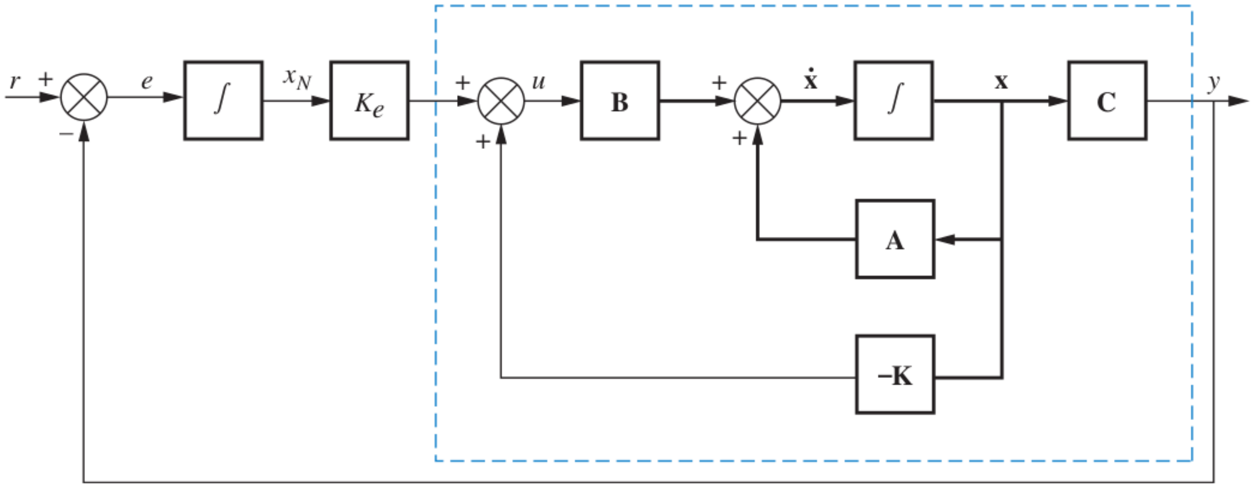
Now calculate O_{Mx}

$$O_{Mx} = \begin{bmatrix} C_x \\ C_x A_x \\ \vdots \\ C_x A_x^{N-1} \end{bmatrix} = O_{Mz}P$$

So, in conclusion:

$$P = O_{Mz}^{-1} O_{Mx} \; L_z = PL_x$$

Integral Control with 0 Steady-State Error



$$U = V - KX$$

$$\frac{(R-Y)}{s}K_e = X_N K_e = V \equiv \frac{Y}{T(s)}$$

So

$$\frac{Y}{R} = \frac{K_e \frac{T(s)}{s}}{1 + K_e \frac{T(s)}{s}}$$

Then

$$\begin{aligned} e_{ss} &= \lim_{s \to 0+} sR(s)(1 - \frac{Y(s)}{R(s)}) \\ &= \lim_{s \to 0+} \frac{1}{1 + K_e \frac{T(s)}{s}} \\ &= \lim_{s \to 0+} \frac{s}{s + K_e T(s)} = 0 \end{aligned}$$

since

$$\dot{x}_N = R - Y = R - CX = [-C \; 0] \begin{bmatrix} X \\ x_N \end{bmatrix} + R$$

so

$$\begin{bmatrix} \dot{X} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} X \\ x_N \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} U + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R$$

since

$$U = K_e x_N - KX = [-K \; K_e] \begin{bmatrix} X \\ x_N \end{bmatrix}$$

we obtain

$$\begin{bmatrix} \dot{X} \\ \dot{x}_N \end{bmatrix} = (\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} [-K \; K_e]) \begin{bmatrix} X \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R = \begin{bmatrix} A - BK & BK_e \\ -C & 0 \end{bmatrix} \begin{bmatrix} X \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R$$

Why zero of T(s) Not change with Controller

we know

$$G(s) = \frac{C \text{adj}(sI - A)B}{|sI - A|}$$
$$T(s) = \frac{C \text{adj}(sI - A + BK)B}{|sI - A + BK|}$$

why the numurator of G(S), T(s) is the same, because $\forall C$, so must prove

$$\text{adj}(sI - A)B = \text{adj}(sI - A + BK)B$$

that is mean $\forall A$ (replace sI-A with A)

$$\text{adj}(A)B = \text{adj}(A + BK)B$$

lemma: Cramer's Rule

for $AX = B$, where

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_N \end{bmatrix}$$

We have

$$X = A^{-1}B = \frac{\mathrm{adj}(A)B}{|A|} = \frac{|A \overset{i}{\leftarrow} B| \text{for } x_i}{|A|}$$

$$x_i|A| = \mathrm{adj}(A)B \quad \text{ith element} = |A \overset{i}{\leftarrow} B|$$

Here

$$A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$$

$$\left(A^i \leftarrow B\right) \overset{\text{def}}{=} (\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{i-1} \quad B \quad \mathbf{a}_{i+1} \quad \cdots \quad \mathbf{a}_n)$$

proof

$$\begin{gathered}\mathrm{adj}(A+BK)B \quad \text{ith element} = |(A+BK) \overset{i}{\leftarrow} B| \\ = |\mathbf{a}_1 + k_1 B \quad \cdots \quad \mathbf{a}_{i-1} + k_{i-1}B \quad B \quad \cdots \quad \mathbf{a}_N + k_N B| \\ = |\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{i-1} \quad B \quad \cdots \quad \mathbf{a}_N| \\ = |A \overset{i}{\leftarrow} B| = \mathrm{adj}(A)B \quad \text{ith element}\end{gathered}$$

Another rule $|A+BK| = |A| + K\mathrm{adj}(A)B$

$$\begin{gathered}|A+BK| = |\mathbf{a}_1 + k_1 B \quad \cdots \quad \mathbf{a}_i + k_i B \quad \cdots \quad \mathbf{a}_N + k_N B| \\ = |\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{i-1} \quad \cdots \quad \mathbf{a}_N| \\ + \sum_i k_i \, |\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{i-1} \quad B \quad \cdots \quad \mathbf{a}_N| \\ = |A| + \sum_i k_i \, [\mathrm{adj}(A)B \quad \text{ith element}] \\ = |A| + K\mathrm{adj}(A)B\end{gathered}$$

Conclusion

$$\begin{gathered}G(s) = \frac{C\mathrm{adj}(sI-A)B}{|sI-A|} = \frac{N(s)}{D_1(s)} \\ T(s) = \frac{C\mathrm{adj}(sI-A+BK)B}{|sI-A+BK|} \\ = \frac{C\mathrm{adj}(sI-A)B}{|sI-A| + K\mathrm{adj}(sI-A)B} = \frac{N(s)}{D_2(s)}\end{gathered}$$

if we introduce K_e

$$
\begin{aligned}
\frac{Y(s)}{R(s)} \equiv T'(s) &= \frac{K_e \frac{T(s)}{s}}{1 + K_e \frac{T(s)}{s}} \\
&= \frac{K_e N(s)}{s D_2(s) + K_e N(s)}
\end{aligned}
$$

even though we have introduced K and K_e, zeros of $T(s)$ wouldn't be changed