Peak time T_p, Settling time T_s, Overshoot %

$$\begin{aligned}G(s) &= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \\ &= \frac{\omega_n^2}{(s+\zeta\omega_n+j\sqrt{1-\zeta^2}\omega_n)(s+\zeta\omega_n-j\sqrt{1-\zeta^2}\omega_n)} \\ &= \frac{A}{s+\zeta\omega_n+j\sqrt{1-\zeta^2}\omega_n} - \frac{A}{s+\zeta\omega_n-j\sqrt{1-\zeta^2}\omega_n}\end{aligned}$$

Where $-2Aj\sqrt{1-\zeta^2}\omega_n = \omega_n^2 \Leftrightarrow A = \frac{j\omega_n}{2\sqrt{1-\zeta^2}}$. So

$$\begin{aligned}G(s) &= \frac{j\omega_n}{2\sqrt{1-\zeta^2}}\left[\frac{1}{s+\zeta\omega_n+j\sqrt{1-\zeta^2}\omega_n} - \frac{1}{s+\zeta\omega_n-j\sqrt{1-\zeta^2}\omega_n}\right] \\ g(t) &= \frac{j\omega_n}{2\sqrt{1-\zeta^2}}[e^{-(\zeta\omega_n+j\sqrt{1-\zeta^2}\omega_n)t} - e^{-(\zeta\omega_n-j\sqrt{1-\zeta^2}\omega_n)t}] \\ &= \frac{j\omega_n}{2\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}[-2j\sin(\sqrt{1-\zeta^2}\omega_n t)] \\ &= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\omega_n t)\end{aligned}$$

So, the output $y(t)$ with $y(0_-) = 0$ would be

$$\begin{aligned}Y(s) = \frac{1}{s}G(s) \Leftrightarrow y(t) = u(t) * g(t) &= \int_{0_-}^t g(\tau)d\tau \\ \mathbb{L}[y'(t)] = sY(s) - y(0_-) = sY(s) &= G(s) \\ y'(t) = g(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}&\sin(\sqrt{1-\zeta^2}\omega_n t)\end{aligned}$$

Consider such an integral, where $\tan(\phi) = \frac{b}{a}$

$$\begin{aligned}\int_0^t e^{-a\tau}\sin(b\tau)d\tau = \int_0^t \mathrm{Im}\left[e^{(-a+jb)\tau}\right]d\tau &= \mathrm{Im}\left[\int_0^t e^{(-a+jb)\tau}d\tau\right] \\ &= \mathrm{Im}\left[\frac{a+jb}{a^2+b^2}[1-e^{-at}\cos(bt)-je^{-at}\sin(bt)]\right] \\ &= \frac{1}{a^2+b^2}[-ae^{-at}\sin(bt)+b-be^{-at}\cos(bt)] \\ &= \frac{b}{a^2+b^2} - \frac{e^{-at}}{a^2+b^2}\sqrt{a^2+b^2}\sin(bt+\phi)\end{aligned}$$

substitute $a, b \leftarrow \zeta\omega_n, \sqrt{1-\zeta^2}\omega_n$, we can derive the expression for $y(t)$

$$
\begin{aligned}
y(t) = \int_{0_-}^{t} g(\tau)d\tau &= \frac{\omega_n}{\sqrt{1-\zeta^2}} \int_{0_-}^{t} e^{-\zeta\omega_n\tau}\sin(\sqrt{1-\zeta^2}\omega_n\tau)d\tau \\
&= \frac{\omega_n}{\sqrt{1-\zeta^2}}\left[\frac{\sqrt{1-\zeta^2}\omega_n}{\omega_n^2} - \frac{e^{-\zeta\omega_n t}}{\omega_n}\sin(\sqrt{1-\zeta^2}\omega_n t + \phi)\right] \\
&= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\sqrt{1-\zeta^2}\omega_n t + \phi)
\end{aligned}
$$

where $\tan(\phi) = \frac{b}{a} = \frac{\sqrt{1-\zeta^2}}{\zeta}, \sin(\phi) = \sqrt{1-\zeta^2}, \cos(\phi) = \zeta$

Calculate T_s

$$
T_s = \frac{4}{\zeta\omega_n} \Leftrightarrow e^{-\zeta\omega_n T_s} = e^{-4} \approx 0.02
$$

Calculate overshoot $\%$ and T_p

by let $y'(t)|_{t=T_p} = 0$

$$
\begin{aligned}
y'(t)|_{t=T_p} &= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T_p}\sin(\sqrt{1-\zeta^2}\omega_n T_p) = 0 \\
\sqrt{1-\zeta^2}\omega_n T_p &= \pi \Leftrightarrow T_p = \frac{\pi}{\sqrt{1-\zeta^2}\omega_n}
\end{aligned}
$$

consider the overshoot $\%$ at $t = T_p$

$$
\begin{aligned}
1 + \% \equiv y(t)|_{t=T_p} &= 1 - \frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\sin(\sqrt{1-\zeta^2}\omega_n T_p + \phi) \\
&= 1 - \frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\sin(\pi + \phi) = 1 + \frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\sqrt{1-\zeta^2} \\
&= 1 + e^{-\zeta\omega_n T_p} = 1 + e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}}
\end{aligned}
$$

So, we find the relationship between the overshoot $\%$ and ζ

$$
\begin{aligned}
\% = e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}} &\Leftrightarrow -\frac{\ln(\%)}{\pi} = \frac{\zeta}{\sqrt{1-\zeta^2}} \\
\frac{-\ln(\%)}{\sqrt{\pi^2 + \ln^2(\%)}} &= \zeta
\end{aligned}
$$

Summary

$$
\begin{aligned}
\zeta &= \frac{-\ln(\%)}{\sqrt{\ln(\%)^2 + \pi^2}} \\
T_s &= \frac{4}{\zeta\omega_n} \\
T_p &= \frac{\pi}{\sqrt{1-\zeta^2}\omega_n}
\end{aligned}
$$

where T_s represents settling time, T_p represents peak time. consider $s^2 + 2\zeta\omega_n s + \omega_n^2$, we can determine the coefficients for $s, 1$ by

$$
\begin{aligned}
2\zeta\omega_n &= \frac{8}{T_s} = \frac{-2\ln(\%)}{T_p} \\
\omega_n^2 &= \frac{16(\ln(\%)^2 + \pi^2)}{\ln(\%)^2 T_s^2} = \frac{\ln(\%)^2 + \pi^2}{T_p^2}
\end{aligned}
$$